

# DERIVED STRING TOPOLOGY AND THE EILENBERG-MOORE SPECTRAL SEQUENCE

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**ABSTRACT.** The loop (co)products on a simply-connected Gorenstein space are described in terms of the torsion and extension functors by developing string topology in appropriate derived categories. As a consequence, it is proved that the composite (*the loop product*)  $\circ$  (*the loop coproduct*) is trivial for a Poincaré duality space. Moreover, we show that the Eilenberg-Moore spectral sequence converging to the loop homology of a Gorenstein space admits a multiplication and a comultiplication with shifted degree which are compatible with the loop product and the loop coproduct of its target, respectively. Especially, in case of a Poincaré duality space, we have a new spectral sequence converging to the Chas-Sullivan loop homology. Our discussions on the loop product are generalized to the case of a relative loop space.

## 1. INTRODUCTION

There are several spectral sequences concerning main players in string topology [8, 5, 31, 44, 25]. Cohen, Jones and Yan [8] have constructed a loop algebra spectral sequence which is of the Leray-Serre type. The Moore spectral sequence converging to the Hochschild cohomology ring of a differential graded algebra is endowed with an algebra structure [17] and moreover a Batalin-Vilkovisky algebra structure [25], which are compatible with such a structure of the target. Very recently, Shamir [44] has constructed a Leray-Serre type spectral sequence converging to the Hochschild cohomology ring of a differential graded algebra. Then one might expect that the Eilenberg-Moore spectral sequence (EMSS), which converges to the loop homology of a closed oriented manifold and of a more general Gorenstein space, enjoys a multiplicative structure corresponding to the loop product.

The class of Gorenstein spaces contains Poincaré duality spaces, for example closed oriented manifolds, and Borel constructions, in particular, the classifying spaces of connected Lie groups; see [10, 41, 27]. In [15], Félix and Thomas develop string topology on Gorenstein spaces. As seen in string topology, the shriek map (the wrong way map) plays an important role when defining string operations. Such a map for a Gorenstein space appears in an appropriate derived category. Thus we

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can discuss string topology due to Chas and Sullivan in the more general setting with cofibrant replacements of the singular cochains on spaces.

In the remainder of this section, our main results are surveyed. We describe explicitly the loop (co)products for a Gorenstein space in terms of the differential torsion product and the extension functors; see Theorems 2.3, 2.5 and 2.14. The key idea of the consideration comes from the general setting in [15] for defining string operations mentioned above. Thus our description of the loop (co)product with derived functors fits *derived string topology*, namely the framework of string topology due to Félix and Thomas. Indeed, according to expectation, the full descriptions of the products with derived functors permits us to give the EMSS (co)multiplicative structures which are compatible with the dual to the loop (co)products of its target; see Theorem 2.8.

By dualizing the EMSS, we obtain a new spectral sequence converging to the Chas-Sullivan loop homology with coefficients in a field  $\mathbb{K}$  of a simply-connected Gorenstein space  $M$  with finite dimensional cohomology, in consequence a Poincaré duality space. We observe that the  $E_2$ -term of the dual EMSS is represented by the Hochschild cohomology ring of  $H^*(M; \mathbb{K})$ ; see Theorems 2.11 and 5.1. It is conjectured that there is an isomorphism of graded algebras between the loop homology of  $M$  and the Hochschild cohomology of the singular cochains on  $M$ . But over  $\mathbb{F}_p$ , even in the case of a simply-connected closed orientable manifold, there is no complete written proof of such an isomorphism of algebras (See [17, p. 237] for details). Anyway, even if we assume such isomorphism, it is not clear that the spectral sequence obtained by filtering Hochschild cohomology is isomorphic to the dual EMSS although these two spectral sequences have the same  $E_2$  and  $E_\infty$ -term. It is worth stressing that the EMSS in Theorem 2.8 is applicable to each space in the more wide class of Gorenstein spaces and is moreover endowed with both the loop product and the loop coproduct. Let  $N$  be a simply-connected space whose cohomology is of finite dimension and is generated by a single element. Then explicit calculations of the dual EMSS yields that the loop homology of  $N$  is isomorphic to the Hochschild cohomology of  $H^*(N; \mathbb{K})$  as an algebra; see Theorem 7.1. This illustrates computability of our spectral sequence in Theorem 2.11.

With the aid of the torsion functor descriptions of the loop (co)products, we see that the composite (*the loop product*)  $\circ$  (*the loop coproduct*) is trivial for a simply-connected Poincaré duality space; see Theorem 2.13. Therefore, the same argument as in the proof of [46, Theorem A] deduces that if string operations on a Poincaré duality space gives rise to a 2-dimensional TQFT, then all operations associated to surfaces of genus at least one vanish. For a more general Gorenstein space, an obstruction for the composite to be trivial can be found in a hom-set, namely the extension functor, in an appropriate derived category; see Remark 4.5. This small but significant result also asserts an advantage of derived string topology.

## 2. DERIVED STRING TOPOLOGY AND MAIN RESULTS

The goal of this section is to state our results in detail. The proofs are found in Sections 3 to 10.

We begin by recalling the most prominent result on shriek maps due to Félix and Thomas, which supplies string topology with many homological and homotopical algebraic tools. Let  $\mathbb{K}$  be a field of arbitrary characteristic. In what follows, we denote by  $C^*(M)$  and  $H^*(M)$  the normalized singular cochain algebra of a space

$M$  with coefficients in  $\mathbb{K}$  and its cohomology, respectively. For a differential graded algebra  $A$ , let  $D(\text{Mod-}A)$  and  $D(A\text{-Mod})$  be the derived categories of right  $A$ -modules and left  $A$ -modules, respectively. Unless otherwise explicitly stated, it is assumed that a space has the homotopy type of a CW-complex whose homology with coefficients in an underlying field is of finite type.

Consider a pull-back diagram  $\mathcal{F}$ :

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{f} & M \end{array}$$

in which  $p$  is a fibration over a simply-connected Poincaré duality space  $M$  of dimension  $m$  with the fundamental class  $\omega_M$  and  $N$  is a Poincaré duality space of dimension  $n$  with the fundamental class  $\omega_N$ .

**Theorem 2.1.** ([29],[15, Theorems 1 and 2]) *With the notation above there exist unique elements*

$$f^! \in \text{Ext}_{C^*(M)}^{m-n}(C^*(N), C^*(M)) \quad \text{and} \quad g^! \in \text{Ext}_{C^*(E)}^{m-n}(C^*(X), C^*(E))$$

such that  $H^*(f^!)(\omega_N) = \omega_M$  and in  $D(\text{Mod-}C^*(M))$ , the following diagram is commutative

$$\begin{array}{ccc} C^*(X) & \xrightarrow{g^!} & C^{*+m-n}(E) \\ q^* \uparrow & & \uparrow p^* \\ C^*(N) & \xrightarrow{f^!} & C^{*+m-n}(M). \end{array}$$

Let  $A$  be a differential graded augmented algebra over  $\mathbb{K}$ . We call  $A$  a *Gorenstein algebra* of dimension  $m$  if

$$\dim \text{Ext}_A^*(\mathbb{K}, A) = \begin{cases} 0 & \text{if } * \neq m, \\ 1 & \text{if } * = m. \end{cases}$$

A path-connected space  $M$  is called a  $\mathbb{K}$ -*Gorenstein space* (simply, Gorenstein space) of dimension  $m$  if the normalized singular cochain algebra  $C^*(M)$  with coefficients in  $\mathbb{K}$  is a Gorenstein algebra of dimension  $m$ . We write  $\dim M$  for the dimension  $m$ .

The result [10, Theorem 3.1] yields that a simply-connected Poincaré duality space, for example a simply-connected closed orientable manifold, is Gorenstein. The classifying space  $BG$  of connected Lie group  $G$  and the Borel construction  $EG \times_G M$  for a simply-connected Gorenstein space  $M$  with  $\dim H^*(M; \mathbb{K}) < \infty$  on which  $G$  acts are also examples of Gorenstein spaces; see [10, 41, 27]. Observe that, for a closed oriented manifold  $M$ ,  $\dim M$  coincides with the ordinary dimension of  $M$  and that for the classifying space  $BG$  of a connected Lie group,  $\dim BG = -\dim G$ . Thus the dimensions of Gorenstein spaces may become negative.

The following theorem enables us to generalize the above result concerning shriek maps on a Poincaré duality space to that on a Gorenstein space.

**Theorem 2.2.** ([15, Theorem 12]) *Let  $X$  be a simply-connected  $\mathbb{K}$ -Gorenstein space of dimension  $m$  whose cohomology with coefficients in  $\mathbb{K}$  is of finite type. Then*

$$\text{Ext}_{C^*(X^n)}^*(C^*(X), C^*(X^n)) \cong H^{*-n(n-1)m}(X),$$

where  $C^*(X)$  is considered a  $C^*(X^n)$ -module via the diagonal map  $\Delta : X \rightarrow X^n$ .

We denote by  $\Delta^!$  the map in  $D(\text{Mod-}C^*(X^n))$  which corresponds to a generator of  $\text{Ext}_{C^*(X^n)}^{(n-1)m}(C^*(X), C^*(X^n)) \cong H^0(X)$ . Then, for a Gorenstein space  $X$  of dimension  $m$  and a fibre square

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{\Delta} & X^n, \end{array}$$

there exists a unique map  $g^!$  in  $\text{Ext}_{C^*(E)}^{(n-1)m}(C^*(E'), C^*(E))$  which fits into the commutative diagram in  $D(\text{Mod-}C^*(X^n))$

$$\begin{array}{ccc} C^*(E') & \xrightarrow{g^!} & C^*(E) \\ (p')^* \uparrow & & \uparrow p^* \\ C^*(X) & \xrightarrow{\Delta^!} & C^*(X^n). \end{array}$$

We remark that the result follows from the same proof as that of Theorem 2.1.

Let  $K \xleftarrow{f} A \xrightarrow{g} L$  be a diagram in the category of differential graded algebras (henceforth called DGA's). We consider  $K$  and  $L$  right and left modules over  $A$  via maps  $f$  and  $g$ , respectively. Then the differential torsion product  $\text{Tor}_A(K, L)$  is denoted by  $\text{Tor}_A(K, L)_{f,g}$  when the actions are emphasized.

We recall here the Eilenberg-Moore map. Consider the pull-back diagram  $\mathcal{F}$  mentioned above, in which  $p$  is a fibration and  $M$  is a simply-connected space. Let  $\varepsilon : F \rightarrow C^*(E)$  be a left semi-free resolution of  $C^*(E)$  in  $C^*(M)\text{-Mod}$  the category of left  $C^*(M)$ -modules. Then the Eilenberg-Moore map

$$EM : \text{Tor}_{C^*(M)}^*(C^*(N), C^*(E))_{f^*, p^*} = H(C^*(N) \otimes_{C^*(M)} F) \longrightarrow H^*(X)$$

is defined by  $EM(x \otimes_{C^*(M)} u) = q^*(x) \smile (g^*\varepsilon(u))$  for  $x \otimes_{C^*(M)} u \in C^*(N) \otimes_{C^*(M)} F$ . Observe that in the same way, we can define the Eilenberg-Moore map by using a semi-free resolution of  $C^*(N)$  as a right  $C^*(M)$ -module. We see that the map  $EM$  is an isomorphism of graded algebras with respect to the cup products; see [20] for example. In particular, for a simply-connected space  $M$ , consider the commutative diagram,

$$\begin{array}{ccccc} LM & \xrightarrow{\quad} & M^I & \xleftarrow[\simeq]{\sigma} & M \\ ev_0 \downarrow & & p=(ev_0, ev_1) \downarrow & & \swarrow \Delta \\ M & \xrightarrow{\quad} & M \times M & & \end{array}$$

where  $ev_i$  stands for the evaluation map at  $i$  and  $\sigma : M \xrightarrow{\simeq} M^I$  for the inclusion of the constant paths. We then obtain the composite  $EM'$ :

$$H^*(LM) \xleftarrow[\simeq]{EM} \text{Tor}_{C^*(M \times 2)}^*(C^*M, C^*M^I)_{\Delta^*, p^*} \xrightarrow[\simeq]{\text{Tor}_1(1, \sigma^*)} \text{Tor}_{C^*(M \times 2)}^*(C^*M, C^*M)_{\Delta^*, \Delta^*}.$$

Our first result states that the torsion functor  $\text{Tor}_{C^*(M \times 2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}$  admits (co)products which are compatible with  $EM'$ .

In order to describe such a result, we first recall the definition of the loop product on a simply-connected Gorenstein space. Consider the diagram

$$(2.1) \quad \begin{array}{ccccc} LM & \xleftarrow{\text{Comp}} & LM \times_M LM & \xrightarrow{q} & LM \times LM \\ ev_0 \downarrow & & \downarrow & & \downarrow (ev_0, ev_1) \\ M & \xlongequal{\quad} & M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where the right-hand square is the pull-back of the diagonal map  $\Delta$ ,  $q$  is the inclusion and  $\text{Comp}$  denotes the concatenation of loops. By definition the composite

$$q^! \circ (\text{Comp})^* : C^*(LM) \rightarrow C^*(LM \times_M LM) \rightarrow C^*(LM \times LM)$$

induces the dual to the loop product  $Dlp$  on  $H^*(LM)$ ; see [15, Introduction]. We see that  $C^*(LM)$  and  $C^*(LM \times LM)$  are  $C^*(M \times M)$ -modules via the map  $ev_0 \circ \Delta$  and  $(ev_0, ev_1)$ , respectively. Moreover since  $q^!$  is a morphism of  $C^*(M \times M)$ -modules, it follows that so is  $q^! \circ (\text{Comp})^*$ . The proof of Theorem 2.1 states that the map  $q^!$  is obtained extending the shriek map  $\Delta^!$ , which is first given, in the derived category  $D(\text{Mod-}C^*(M \times M))$ . This fact allows us to formulate  $q^!$  in terms of differential torsion functors.

**Theorem 2.3.** *Let  $M$  be a simply-connected Gorenstein space of dimension  $m$ . Consider the comultiplication given by the composite*

$$\begin{array}{c} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*} \xrightarrow{\text{Tor}_{p_{13}^*}^{(1,1)}} \text{Tor}_{C^*(M^3)}^*(C^*(M), C^*(M))_{((1 \times \Delta) \circ \Delta)^*, ((1 \times \Delta) \circ \Delta)^*} \\ \cong \uparrow \text{Tor}_{(1 \times \Delta \times 1)^*}(1, \Delta^*) \\ \text{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^2))_{(\Delta^2 \circ \Delta)^*, \Delta^{2*}} \\ \downarrow \text{Tor}_1(\Delta^!, 1) \\ \left( \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))^{\otimes 2}_{\Delta^*, \Delta^*} \right)^{*+m} \xrightarrow[\tilde{\top}]{} \text{Tor}_{C^*(M^4)}^{*+m}(C^*(M^2), C^*(M^2))_{\Delta^{2*}, \Delta^{2*}}. \end{array}$$

See Remark 2.4 below for the definition of  $\tilde{\top}$ . Then the composite  $EM'$  :

$$H^*(LM) \xrightarrow[\cong]{EM^{-1}} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I))_{\Delta^*, p^*} \xrightarrow[\cong]{\text{Tor}_1(1, \sigma^*)} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}$$

is an isomorphism which respects the dual to the loop product  $Dlp$  and the comultiplication defined here.

*Remark 2.4.* The isomorphism  $\tilde{\top}$  in Theorem 2.3 is the canonical map defined by [20, p. 26] or by [33, p. 255] as the composite

$$\begin{array}{ccc} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))^{\otimes 2} & \xrightarrow{\top} & \text{Tor}_{C^*(M^2)^{\otimes 2}}^*(C^*(M)^{\otimes 2}, C^*(M)^{\otimes 2}) \\ & & \downarrow \text{Tor}_\gamma(\gamma, \gamma) \\ \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2)) & \xrightarrow[\text{Tor}_{EZ^\vee}(EZ^\vee, EZ^\vee)]{\cong} & \text{Tor}_{(C^*(M^2)^{\otimes 2})^\vee}^*((C^*(M)^{\otimes 2})^\vee, (C^*(M)^{\otimes 2})^\vee) \end{array}$$

where  $\top$  is the  $\top$ -product of Cartan-Eilenberg [3, XI. Proposition 1.2.1] or [32, VIII.Theorem 2.1],  $EZ : C_*(M)^{\otimes 2} \xrightarrow{\cong} C_*(M^2)$  denotes the Eilenberg-Zilber quasi-isomorphism and  $\gamma : \text{Hom}(C_*(M), \mathbb{K})^{\otimes 2} \rightarrow \text{Hom}(C_*(M)^{\otimes 2}, \mathbb{K})$  is the canonical map.

It is worth mentioning that this theorem gives an intriguing decomposition of the cup product on the Hochschild cohomology of a commutative algebra; see Lemma 10.3 below.

The loop coproduct on a Gorenstein space is also interpreted in terms of torsion products. In order to recall the loop coproduct, we consider the commutative diagram

$$(2.2) \quad \begin{array}{ccccc} LM \times LM & \xleftarrow{q} & LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\ & & \downarrow & & \downarrow l \\ & & M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where  $l : LM \rightarrow M \times M$  is a map defined by  $l(\gamma) = (\gamma(0), \gamma(\frac{1}{2}))$ . By definition, the composite

$$\text{Comp}^! \circ q^* : C^*(LM \times LM) \rightarrow C^*(LM \times_M LM) \rightarrow C^*(LM)$$

induces the dual to the loop coproduct  $Dlcop$  on  $H^*(LM)$ .

Note that we apply Theorem 2.1 to (2.2) in defining the loop coproduct. On the other hand, applying Theorem 2.1 to the diagram (2.1), the loop product is defined.

**Theorem 2.5.** *Let  $M$  be a simply-connected Gorenstein space of dimension  $m$ . Consider the multiplication defined by the composite*

$$\begin{array}{ccc} \left( \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*} \right)^{\otimes 2} & \xrightarrow[\tilde{\top}]{} & \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2))_{\Delta^{2*}, \Delta^{2*}} \\ \downarrow & & \downarrow \text{Tor}_1(\Delta^*, 1) \\ & & \text{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^2))_{(\Delta^2 \circ \Delta)^*, \Delta^{2*}} \\ \downarrow & & \downarrow \text{Tor}_1(\Delta^!, 1) \\ \text{Tor}_{C^*(M^2)}^{*+m}(C^*(M), C^*(M))_{\Delta^*, \Delta^*} & \xleftarrow[\text{Tor}_{\alpha^*}(\Delta^*, \Delta^*)]{} & \text{Tor}_{C^*(M^4)}^{*+m}(C^*(M^2), C^*(M^2))_{\gamma'^*, \Delta^{2*}} \end{array}$$

where the maps  $\alpha : M^2 \rightarrow M^4$  and  $\gamma' : M^2 \rightarrow M^4$  are defined by  $\alpha(x, y) = (x, y, y, y)$  and  $\gamma'(x, y) = (x, y, y, x)$ . See remark 2.4 above for the definition of  $\tilde{\top}$ . Then the composite  $EM'$  :

$$H^*(LM) \xrightarrow[\cong]{EM'} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I))_{\Delta^*, p^*} \xrightarrow[\cong]{\text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}} \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}$$

is an isomorphism respects the dual to the loop coproduct  $Dlcop$  and the multiplication defined here.

**Remark 2.6.** A relative version of the loop product is also in our interest. Let  $f : N \rightarrow M$  be a map. Then by definition, the relative loop space  $L_f M$  fits into the pull-back diagram

$$\begin{array}{ccc} L_f M & \longrightarrow & M^I \\ \downarrow & & \downarrow (ev_0, ev_1) \\ N & \xrightarrow{(f, f)} & M \times M, \end{array}$$

where  $ev_t$  denotes the evaluation map at  $t$ . We may write  $L_N M$  for the relative loop space  $L_f M$  in case there is no danger of confusion. Suppose further that  $M$  is

simply-connected and has a base point. Let  $N$  be a simply-connected Gorenstein space. Then the diagram

$$\begin{array}{ccccc} L_N M & \xleftarrow{\text{Comp}} & L_N M \times_N L_N M & \xrightarrow{q} & L_N M \times L_N M \\ \downarrow & & \downarrow & & \downarrow (ev_0, ev_1) \\ N & \xrightarrow{\Delta} & N \times N & & \end{array}$$

gives rise to the composite

$$q^! \circ (\text{Comp})^* : C^*(L_N M) \rightarrow C^*(L_N M \times_N L_N M) \rightarrow C^*(L_N M \times L_N M)$$

which, by definition, induces the dual to the relative loop product  $Drlp$  on the cohomology  $H^*(L_N M)$  with degree  $\dim N$ ; see [17, 19] for case that  $N$  is a smooth manifold. Since the diagram above corresponds to the diagram (2.1), the proof of Theorem 2.3 permits one to conclude that  $Drlp$  has also the same description as in Theorem 2.3, where  $C^*(N)$  is put instead of  $C^*(M)$  in the left-hand variables of the torsion functors in the theorem.

As for the loop coproduct, we cannot define its relative version in natural way because of the evaluation map  $l$  of loops at  $\frac{1}{2}$ ; see the diagram (2.2). Indeed the point  $\gamma(\frac{1}{2})$  for a loop  $\gamma$  in  $L_N M$  is not necessarily in  $N$ .

The associativity of  $Dlp$  and  $Dlcop$  on a Gorenstein space is an important issue. We describe here an algebra structure on the shifted homology  $H_{-*+d}(L_N M) = (H^*(L_N M)^\vee)^{*-d}$  of a simply-connected Poincaré duality space  $N$  of dimension  $d$  with a map  $f : N \rightarrow M$  to a simply-connected space.

We define a map  $m : H_*(L_N M) \otimes H_*(L_N M) \rightarrow H_*(L_N M)$  of degree  $d$  by

$$m(a \otimes b) = (-1)^{d(|a|+d)} ((Drlp)^\vee)(a \otimes b)$$

for  $a$  and  $b \in H_*(L_N M)$ ; see [8, sign of Proposition 4]. Moreover, put  $\mathbb{H}_*(L_N M) = H_{*+d}(L_N M)$ . Then we establish the following proposition.

**Proposition 2.7.** *Let  $N$  be a simply-connected Poincaré duality space. Then the shifted homology  $\mathbb{H}_*(L_N M)$  is a unital associative algebra with respect to the product  $m$ . Moreover, if  $M = N$ , then the shifted homology  $\mathbb{H}_*(LM)$  is graded commutative.*

As mentioned below, the loop product on  $L_N M$  is not commutative in general.

We call a bigraded vector space  $V$  a *bimagma* with shifted degree  $(i, j)$  if  $V$  is endowed with a multiplication  $V \otimes V \rightarrow V$  and a comultiplication  $V \rightarrow V \otimes V$  of degree  $(i, j)$ .

Let  $K$  and  $L$  be objects in  $\text{Mod-}A$  and  $A\text{-Mod}$ , respectively. Consider a torsion product of the form  $\text{Tor}_A(K, L)$  which is the homology of the derived tensor product  $K \otimes_A^{\mathbb{L}} L$ . The external degree of the bar resolution of the second variable  $L$  filters the torsion products. Indeed, we can regard the torsion product  $\text{Tor}_A(K, L)$  as the homology  $H(M \otimes_A B(A, A, L))$  with the bar resolution  $B(A, A, L) \rightarrow L$  of  $L$ . Then the filtration  $\mathcal{F} = \{F^p \text{Tor}_A(K, L)\}_{p \leq 0}$  of the torsion product is defined by

$$F^p \text{Tor}_A(K, L) = \text{Im}\{i^* : H(M \otimes_A B^{\leq p}(A, A, L)) \rightarrow \text{Tor}_A(K, L)\}.$$

Thus the filtration  $\mathcal{F} = \{F^p \text{Tor}_{C^*(M^2)}(C^*(M), C^*(M^I))\}_{p \leq 0}$  induces a filtration of  $H^*(LM)$  via the Eilenberg-Moore map for a simply-connected space  $M$ .

By adapting differential torsion functor descriptions of the loop (co)products in Theorems 2.3 and 2.5, we can give the EMSS a bimagma structure.

**Theorem 2.8.** *Let  $M$  be a simply-connected Gorenstein space of dimension  $d$ . Then the Eilenberg-Moore spectral sequence  $\{E_r^{*,*}, d_r\}$  converging to  $H^*(LM; \mathbb{K})$  admits loop (co)products which is compatible with those in the target; that is, each term  $E_r^{*,*}$  is endowed with a comultiplication  $\delta : E_r^{p,q} \rightarrow \bigoplus_{s+s'=p, t+t'=q+d} E_r^{s,t} \otimes E_r^{s',t'}$  and a multiplication  $m : E_r^{s,t} \otimes E_r^{s',t'} \rightarrow E_r^{s+s',t+t'+d}$  which are compatible with differentials in the sense that*

$$\delta d_r = (-1)^d (d_r \otimes 1 \pm 1 \otimes d_r) \delta \quad \text{and} \quad m(d_r \otimes 1 \pm 1 \otimes d_r) = (-1)^d d_r m,$$

*moreover the  $E_\infty$ -term  $E_\infty^{*,*}$  is isomorphic to  $\text{Gr}H^*(LM; \mathbb{K})$  as a bimagma with shifted degree  $(0, m)$ .*

If the dimension of the Gorenstein space is non-positive, unfortunately the loop product and the loop coproduct in the EMSS are trivial and the only information that Theorem 2.8 gives is the following corollary.

**Corollary 2.9.** *Let  $M$  be a simply-connected Gorenstein space of dimension  $d$ . Assume that  $d$  is negative or that  $d$  is null and  $H^*(M)$  is not concentrated in degree 0. Consider the filtration given by the cohomological Eilenberg-Moore spectral sequence converging to  $H^*(LM; \mathbb{K})$ . Then the dual to the loop product and that to the loop coproduct increase both the filtration degree of  $H^*(LM)$  by at least one.*

*Remark 2.10.* Let  $M$  be a simply-connected closed oriented manifold. We can choose a map  $\Delta^! : C^*(M) \rightarrow C^*(M \times M)$  so that  $H(\Delta^!)w_M = w_{M \times M}$ ; that is,  $\Delta^!$  is the usual shriek map in the cochain level. Then the map  $Dlp$  and  $Dlcop$  coincide with the dual to the loop product and to the loop coproduct in the sense of Chas and Sullivan [4], Cohen and Godin [9], respectively. Indeed, this fact follows from the uniqueness of shriek map and the comments in three paragraphs in the end of [15, p. 421]. Thus the Eilenberg-Moore spectral sequence in Theorem 2.8 converges to  $H^*(LM; \mathbb{K})$  as an algebra and a coalgebra.

Let  $M$  be the classifying space  $BG$  of a connected Lie group  $G$ . Since the homotopy fibre of  $\Delta : BG \rightarrow BG \times BG$  in (2.1) and (2.2) is homotopy equivalent to  $G$ , we can choose the shriek map  $\Delta^!$  described in Theorems 2.5 and 2.3 as the integration along the fibre. Thus  $q^!$  also coincides with the integration along the fibre; see [15, Theorems 6 and 13]. This yields that the bimagma structure in  $\text{Gr}H^*(LBG; \mathbb{K})$  is induced by the loop product and coproduct in the sense of Chataur and Menichi [6].

The cohomology of a closed manifold is indeed a Poincaré duality DGA with trivial differential. Thus Theorem 5.3 below allows us to obtain a spectral sequence for computing the Chas-Sullivan loop homology of a closed oriented manifold, more general, of a Poincaré duality space.

The following theorem is the main result of this paper.

**Theorem 2.11.** *Let  $N$  be a simply-connected Gorenstein space of dimension  $d$ . Let  $f : N \rightarrow M$  be a continuous map to a simply-connected space  $M$ . Then the Eilenberg-Moore spectral sequence is a right-half plane cohomological spectral sequence  $\{\mathbb{E}_r^{*,*}, d_r\}$  converging to the Chas-Sullivan loop homology  $\mathbb{H}_*(L_N M)$  as an algebra with*

$$\mathbb{E}_2^{*,*} \cong HH^{*,*}(H^*(M); \mathbb{H}_*(N))$$

as a bigraded algebra; that is, there exists a decreasing filtration  $\{F^p \mathbb{H}_*(L_N M)\}_{p \geq 0}$  of  $\mathbb{H}_*(L_N M)$  such that  $\mathbb{E}_\infty^{*,*} \cong Gr^{*,*} \mathbb{H}_*(L_N M)$  as a bigraded algebra, where

$$Gr^{p,q} \mathbb{H}_*(L_N M) = F^p \mathbb{H}_{-(p+q)}(L_N M) / F^{p+1} \mathbb{H}_{-(p+q)}(L_N M).$$

Here the product on the  $E_2$ -term is the cup product (See Definition 10.1 (1)) induced by

$$\overline{H(\Delta^{!V})} : \mathbb{H}_*(N) \otimes_{H^*(M)} \mathbb{H}_*(N) \rightarrow \mathbb{H}_*(N).$$

Suppose further that  $N$  is a Poincaré duality space. Then the  $E_2$ -term is isomorphic to the Hochschild cohomology  $HH^{*,*}(H^*(M); H^*(N))$  with the cup product as an algebra.

Taking  $N$  to be the point, we obtain the following well-known corollary.

**Corollary 2.12.** (cf. [33, Corollary 7.19]) Let  $M$  be a pointed topological space. Then the Eilenberg-Moore spectral sequence  $E_2^{*,*} = \text{Ext}_{H^*(M)}^{*,*}(\mathbb{K}, \mathbb{K})$  converging to  $H_*(\Omega M)$  is a spectral sequence of algebras with respect to the Pontryagin product.

When  $M = N$  is a closed manifold, Theorem 2.11 has been announced by McClure in [35, Theorem B]. But there is no proof in [35]. Moreover, McClure claimed that when  $M = N$ , the Eilenberg-Moore spectral sequence is a morphism of BV-algebras. We have not yet been able to prove this very interesting claim.

Let  $f : M \rightarrow K(\mathbb{Z}, 2) = BS^1$  be a map from a simply-connected Poincaré duality space. By applying the spectral sequence in Theorem 2.11, we see that

$$\mathbb{H}_*(L_M K(\mathbb{Z}, 2); \mathbb{K}) \cong H^*(M; \mathbb{K}) \otimes \wedge(y)$$

as an algebra, where  $\deg x \otimes y = -\deg x + 1$ ; see Proposition 9.5. The result [42, Proposition 6.1] due to the third author asserts that the relative loop product is not graded commutative in general. On the other hand, the explicit calculation shows that the loop product on  $\mathbb{H}_*(L_M K(\mathbb{Z}, 2); \mathbb{K})$  is graded commutative.

For instance, consider the inclusion  $\mathbb{C}P^n \rightarrow K(\mathbb{Z}, 2)$ . The result above says that the algebra structure of the loop homology  $\mathbb{H}_*(L_{\mathbb{C}P^n} K(\mathbb{Z}, 2))$  is comparatively simple than that of  $L\mathbb{C}P^n$ ; see Theorems 7.7, 7.8 and 7.9.

Since  $K(\mathbb{Z}, 2) = BS^1$  is a Gorenstein space of dimension  $-1$ , the shifted homology  $\mathbb{H}_*(LK(\mathbb{Z}, 2)) = H_{*-1}(LK(\mathbb{Z}, 2))$  is endowed with the loop product as mentioned above. However, results [45, Theorem 4.5 (i)] and [26, Theorem 2.1] assert that the loop product on  $K(\mathbb{Z}, 2)$  is trivial. It seems that the homology invariant, the loop product captures notably variations of the spaces in which whole loops or their stating points move.

We summarize here spectral sequences converging the loop homology and the Hochschild cohomology of the singular cochain on a space, which are mentioned at the beginning of the Introduction.

The homological Leray-Serre type	The cohomological Eilenberg-Moore type
$E_{-p,q}^2 = H^p(M; H_q(\Omega M))$ $\Rightarrow \mathbb{H}_{-p+q}(LM)$ as an algebra, where $M$ is a simply-connected closed oriented manifold; see [8].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M))$ $\Rightarrow \mathbb{H}_{-p-q}(LM)$ as an algebra, where $M$ is a simply-connected Poincaré duality space; see Theorem 2.11.
$E_{p,q}^2 = H^{-p}(M) \otimes \text{Ext}_{C^*(M)}^{-q}(\mathbb{K}, \mathbb{K})$ $\Rightarrow HH^{-p-q}(C^*(M); C^*(M))$ as an algebra, where $M$ is a simply- connected space whose cohomology is locally finite; see [44].	$E_2^{p,q} = HH^{p,q}(H^*(M); H^*(M))$ $\Rightarrow HH^{p+q}(C^*(M); C^*(M))$ as a B-V algebra, where $M$ is a simply- connected Poincaré duality space; see [25].

Observe that each spectral sequence in the table above converges strongly to the target.

It is important to remark that, for a fibration  $N \rightarrow X \rightarrow M$  of closed orientable manifolds, Le Borgne [31] has constructed a spectral sequence converging to the loop homology  $\mathbb{H}_*(LX)$  as an algebra with  $E_2 \cong \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$  under an appropriate assumption; see also [5] for applications of the spectral sequence. We refer the reader to [36] for spectral sequences concerning a generalized homology theory in string topology.

We focus on a global nature of the loop (co)product. Drawing on the torsion functor description of the loop product and the loop coproduct mentioned in Theorems 2.3 and 2.5, we have the following result.

**Theorem 2.13.** *Let  $M$  be a simply-connected Poincaré duality space. Then the composite (the loop product)  $\circ$  (the loop coproduct) is trivial.*

When  $M$  is a connected closed oriented manifold, the triviality of this composite was first proved by Tamanoi [46, Theorem A]. Tamanoi has also shown that this composite is trivial when  $M$  is the classifying space  $BG$  of a connected Lie group  $G$  [45, Theorem 4.4].

We are aware that the description of the loop coproduct in Theorem 2.5 has no *opposite arrow* such as  $\text{Tor}_{(1 \times \Delta \times 1)^*}(1, \Delta^*)$  in Theorem 2.3. This is a key to the proof of Theorem 2.13. Though we have not yet obtained the same result as Theorem 2.13 on a more general Gorenstein space, some obstruction for the composite to be trivial is described in Remark 4.5.

We may describe the loop product in terms of the extension functor.

**Theorem 2.14.** *Let  $M$  be a simply-connected Poincaré duality space. Consider the multiplication defined by the composite*

$$\begin{array}{ccc}
\text{Ext}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}^{\otimes 2} & \xrightarrow[\cong]{\tilde{\vee}} & \text{Ext}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2))_{\Delta^{2*}, \Delta^{2*}} \\
\downarrow & & \downarrow \text{Ext}_1(1, \Delta^*) \\
& & \text{Ext}_{C^*(M^4)}^*(C^*(M^2), C^*(M))_{\Delta^{2*}, (\Delta^2 \circ \Delta)^*} \\
\downarrow & & \uparrow \cong \text{Ext}_{(1 \times \Delta \times 1)^*}(\Delta^*, 1) \\
\text{Ext}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*} & \xleftarrow[\text{Ext}_{p_{13}^*}(1, 1)]{} & \text{Ext}_{C^*(M^3)}^*(C^*(M), C^*(M))_{((1 \times \Delta) \circ \Delta)^*, ((1 \times \Delta) \circ \Delta)^*}.
\end{array}$$

See Remark 2.15 below for the definition of  $\tilde{\vee}$ . The cap with a representative  $\eta$  of the fundamental class  $[M] \in H_m(M)$  gives a quasi-isomorphism of right- $C^*(M)$ -modules of upper degree  $-m$ ,

$$\text{cap}_M : C^*(M) \xrightarrow{\sim} C_{m-*}(M), x \mapsto \eta \cap x.$$

Let  $\Phi : H^{*+m}(LM) \xrightarrow{\cong} \text{Tor}_{C^*(M^{\times 2})}^*(C_*(M), C^*(M))$  be the composite of the isomorphisms

$$\begin{array}{c}
H^{p+m}(LM) \xrightarrow[\cong]{EM^{-1}} \text{Tor}_{C^*(M^2)}^{p+m}(C^*(M), C^*(M^I))_{\Delta^*, p^*} \xrightarrow[\cong]{\text{Tor}_1(1, \sigma^*)} \text{Tor}_{C^*(M^2)}^{p+m}(C^*(M), C^*(M))_{\Delta^*, \Delta^*} \\
\cong \downarrow \text{Tor}_1(\text{cap}_M, 1) \\
\text{Tor}_{C^*(M^2)}^p(C_*(M), C^*(M)).
\end{array}$$

Then the dual of  $\Phi$ ,  $\Phi^\vee : \text{Ext}_{C^*(M^2)}^{-p}(C^*M, C^*M)_{\Delta^*, \Delta^*} \rightarrow H_{p+m}(LM)$  is an isomorphism which respects the multiplication defined here and the loop product.

*Remark 2.15.* The isomorphism  $\tilde{\vee}$  in Theorem 2.14 is the composite

$$\begin{array}{ccc}
 \text{Ext}_{C^*(M^2)}^*(C^*M, C^*M)^{\otimes 2} & \xrightarrow{\vee} & \text{Ext}_{C^*(M^2) \otimes 2}^*(C^*(M)^{\otimes 2}, C^*(M)^{\otimes 2}) \\
 & & \downarrow \text{Ext}_1(1, \gamma) \\
 \text{Ext}_{(C_*(M^2))^{\otimes 2}\vee}^*((C_*(M)^{\otimes 2})^\vee, (C_*(M)^{\otimes 2})^\vee) & \xrightarrow[\cong]{\text{Ext}_{\gamma(\gamma, 1)}} & \text{Ext}_{C^*(M^2) \otimes 2}^*(C^*(M)^{\otimes 2}, (C_*(M)^{\otimes 2})^\vee) \\
 \downarrow \text{Ext}_{EZ\vee}(EZ^\vee, 1) \cong & & \\
 \text{Ext}_{C^*(M^4)}^*(C^*(M^2), (C_*(M)^{\otimes 2})^\vee) & \xleftarrow[\cong]{\text{Ext}_1(1, EZ^\vee)} & \text{Ext}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2))
 \end{array}$$

where  $\vee$  is the  $\vee$ -product of Cartan-Eilenberg [3, XI. Proposition 1.2.3] or [32, VIII.Theorem 4.2],  $EZ : C_*(M)^{\otimes 2} \xrightarrow{\sim} C_*(M^2)$  denotes the Eilenberg-Zilber quasi-isomorphism and  $\gamma : \text{Hom}(C_*(M), \mathbb{K})^{\otimes 2} \rightarrow \text{Hom}(C_*(M)^{\otimes 2}, \mathbb{K})$  is the canonical map.

*Remark 2.16.* We believe that the multiplication on  $\text{Ext}_{C^*(M^2)}^*(C^*(M), C^*(M))_{\Delta^*, \Delta^*}$  defined in Theorem 2.14 coincides with the Yoneda product.

Denote by  $A(M)$  the functorial commutative differential graded algebra  $A_{PL}(M)$ ; see [13, Corollary 10.10]. Let  $\varphi : A(M)^{\otimes 2} \xrightarrow{\sim} A(M^2)$  be the quasi-isomorphism of algebras given by [13, Example 2 p. 142-3]. Remark that the composite  $\Delta^* \circ \varphi$  coincides with the multiplication of  $A(M)$ . Remark also that we have an Eilenberg-Moore isomorphism  $EM$  for the functor  $A(M)$ ; see [13, Theorem 7.10].

Replacing the singular cochains over the rationals  $C^*(M; \mathbb{Q})$  by the commutative algebra  $A_{PL}(M)$  in Theorem 2.3, we obtain the following theorem.

**Theorem 2.17.** (Compare with [16]) Let  $N$  be a simply-connected Gorenstein space of dimension  $m$  and  $N \rightarrow M$  a continuous map. Let  $\Phi$  be the map given by the commutative square

$$\begin{array}{ccc}
 H^{p+m}(A(L_N M)) & \xleftarrow[\cong]{EM} & \text{Tor}_{-p-m}^{A(M^2)}(A(N), A(M^I))_{\Delta^*, p^*} \\
 \downarrow \Phi & & \downarrow \cong \text{Tor}^1(1, \sigma^*) \\
 HH_{-p-m}(A(M), A(N)) & \xrightarrow[\cong]{\text{Tor}^\varphi(1, 1)} & \text{Tor}_{-p-m}^{A(M^2)}(A(N), A(M))_{\Delta^*, \Delta^*}.
 \end{array}$$

Then the dual  $HH^{-p-m}(A(M), A(N)^\vee) \xrightarrow{\Phi^\vee} H_{p+m}(L_N M; \mathbb{Q})$  to  $\Phi$  is an isomorphism of graded algebras with respect to the loop product and the generalized cup product on Hochschild cohomology induced by  $(\Delta_{A(N)})^\vee : A(N)^\vee \otimes A(N)^\vee \rightarrow A(N)^\vee$  (See Example 10.6).

*Remark 2.18.* Such an isomorphism of algebras between Hochschild cohomology and Chas-Sullivan loop space homology was first proved in [16]. But here our isomorphism is explicit since we do not use a Poincaré duality DGA model for  $A(M)$  given by [30]. In fact, as explain in [16], such an isomorphism is an isomorphism of BV-algebras, since  $\Phi$  is compatible with the circle action and Connes boundary

map. Here the BV-algebra on  $HH^*(A(M), A(M))$  is given by [37, Theorem 18 or Proof of Corollary 20].

In the forthcoming paper [26], we discuss the loop (co)products on the classifying space  $BG$  of a Lie group  $G$  by looking at the integration along the fibre ( $Comp$ )<sup>!</sup>:  $H^*(LBG \times_{BG} LBG) \rightarrow H^*(LM)$  of the homotopy fibration  $G \rightarrow LBG \times_{BG} LBG \rightarrow BG$ . In a sequel [27], we intend to investigate duality on extension groups of the (co)chain complexes of spaces. Such discussion enables one to deduce that Noetherian H-spaces are Gorenstein. In adding, the loop homology of a Noetherian H-space is considered.

The rest of this paper is organized as follows. Section 3 is devoted to proving Theorems 2.3, 2.5, 2.8 and Corollary 2.9. Theorem 2.13 is proved in Section 4. In Section 5, we present a prove of Theorem 2.11. We prove Proposition 2.7 and discuss the associativity and commutativity of the loop product on Poincaré duality space in Section 6. In Section 7, by making use of the spectral sequence described in Theorem 2.11, we compute explicitly the Chas-Sullivan loop homology of a Poincaré duality space. Section 8 discusses a method for solving extension problems in the  $E_\infty$ -term of our spectral sequence. The naturality of the spectral sequence described in Theorem 2.11 is discussed in Section 9. Computational examples on the relative loop homology is given in the end of this section. In particular, for a simply-connected Lie group  $G$  containing  $SU(2)$  as a subgroup, we compare the relative loop homology  $\mathbb{H}_*(L_G(G/SU(2))$  with  $\mathbb{H}_*(L(G/SU(2))$  and  $\mathbb{H}_*(LG)$ ; see Proposition 9.6. Section 10 proves Theorem 2.14 and 2.17. In Appendix, shriek maps on Gorenstein spaces are considered and their important properties, which we use in the body of the paper, are described.

Almost results of the loop product on  $\mathbb{H}_*(LM)$  in this paper are generalized to those on the relative loop homology  $\mathbb{H}_*(L_N M)$  when  $N$  is a Poincaré duality space. We shall give comments in an appropriate remark if such a result remains valid while proofs are left to the reader.

### 3. PROOFS OF THEOREMS 2.3, 2.5 AND 2.8

In order to prove Theorem 2.3, we consider two commutative diagrams

$$(3.1) \quad \begin{array}{ccccc} & LM \times LM & \xrightarrow{i} & M^I \times M^I & \\ q \nearrow & \downarrow p \times p & & \swarrow \tilde{q} & \\ LM \times_M LM & \xrightarrow{j} & M^I \times_M M^I & & \downarrow p \times p = p^2 \\ \downarrow & \downarrow \Delta & \downarrow ev_0, ev_1, ev_1 = u & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M & \xrightarrow{\Delta \times \Delta} & M^4 \\ & & \downarrow & & \searrow 1 \times \Delta \times 1 = w \\ & & M & \xrightarrow{(1 \times \Delta) \circ \Delta = v} & M^3 \end{array}$$

and

$$(3.2) \quad \begin{array}{ccccc} & LM \times_M LM & \xrightarrow{j} & M^I \times_M M^I & \\ Comp \nearrow & \downarrow k & & \swarrow Comp=c & \\ LM & \xrightarrow{p} & M^I & \xleftarrow{(ev_0, ev_1 = ev_0, ev_1) = u} & \\ \downarrow & \downarrow (1 \times \Delta) \circ \Delta & \downarrow p & & \downarrow p_{13} \\ M & \xrightarrow{\Delta} & M \times M & \xrightarrow{p_{13}} & M^3 \end{array}$$

in which front and back squares are pull-back diagrams. Observe that the left and right hand side squares in (3.1) are also pull-back diagrams. Here  $\Delta$  and  $k$  denote the diagonal map and the inclusion, respectively. Moreover  $p_{13} : M^3 \rightarrow M^2$  is the projection defined by  $p_{13}(x, y, z) = (x, z)$  and  $Comp : LM \times_M LM \rightarrow LM$  stands for the concatenation of loops. The cube (3.2) first appeared in [16, p. 320].

*Proof of Theorem 2.3.* Consider the diagram  
(3.3)

$$\begin{array}{ccccc}
& & \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I))_{\Delta^*, p^*} & \xrightarrow{\text{Tor}_{p_{13}^*}^{*(1,c^*)}} & \text{Tor}_{C^*(M^3)}^*(C^*(M), C^*(M^I \times_M M^I))_{v^*, u^*} \\
& & EM \downarrow \cong & & \cong \uparrow \text{Tor}_{w^*}(1, \tilde{q}^*) \\
& & H^*(LM) & \xrightarrow{\cong} & \\
& & Comp^* \downarrow & \nearrow EM_1 & \cong \uparrow \text{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^I \times M^I))_{(wv)^*, p^{2*}} \\
Dlp & \curvearrowright & H^*(LM \times_M LM) & \xleftarrow{\cong} & \\
& & H^*(q^!) \downarrow & \swarrow EM_2 & \downarrow \text{Tor}_1(\Delta^!, 1) \\
& & H^{*+m}(LM \times LM) & \xleftarrow{\cong} & \\
& & EM_3 \nearrow & &
\end{array}$$

where  $EM$  and  $EM_i$  denote the Eilenberg-Moore maps.

The diagram (3.2) is a morphism of pull-backs from the back face to the front face. Therefore the naturality of the Eilenberg-Moore map yields that the upper-left triangle is commutative.

We now consider the front square and the right-hand side square in the diagram (3.1). The squares are pull-back diagrams and hence we have a large pull-back one connecting them. Therefore the naturality of the Eilenberg-Moore map shows that the triangle in the center of the diagram (3.3) is commutative. Thus it follows that the map  $\text{Tor}_{w^*}(1, \tilde{q}^*)$  is an isomorphism.

Let  $\varepsilon : \mathbb{B} \xrightarrow{\sim} C^*(M)$  be a right  $C^*(M^2)$ -semifree resolution of  $C^*(M)$  and  $\varepsilon' : \mathbb{B}' \xrightarrow{\sim} C^*(M^I \times M^I)$  be a left  $C^*(M^4)$ -semifree resolution of  $C^*(M^I \times M^I)$ . We have a commutative diagram

$$\begin{array}{ccc}
(3.4) & & \\
& & C^*(LM \times_M LM) & & C^*(LM \times LM) \\
& & EM \uparrow \cong & & \parallel \\
& & \mathbb{B} \otimes_{C^*(M^2)} C^*(LM \times LM) & \xrightarrow{\Delta^! \otimes 1_{C^*(LM \times LM)}} & C^*(M^2) \otimes_{C^*(M^2)} C^*(LM \times LM) \\
& & \cong \uparrow EM'_2 & & \cong \uparrow 1 \otimes EM'_3 \\
& & 1 \otimes EM'_3 \uparrow \cong & & \\
& & \mathbb{B} \otimes_{C^*(M^2)} (C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}') & \xrightarrow{\Delta^! \otimes 1} & C^*(M^2) \otimes_{C^*(M^2)} (C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}') \\
& & \cong \uparrow \varepsilon \otimes 1 & & \parallel \\
& & C^*(M) \otimes_{C^*(M^2)} (C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}') & & C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}', \\
& & \curvearrowright & &
\end{array}$$

where  $EM'_i$  is the chain map which induces the isomorphism  $EM_i$ . The commutativity of the square and of the left-hand side in (3.4) implies that of the lower square in (3.3). Indeed, observe that  $H^*(q^!) = H^*(\Delta^! \otimes 1_{C^*(LM \times LM)}) \circ H^*(EM)^{-1}$  and  $\text{Tor}_1(\Delta^!, 1) = H^*(\Delta^! \otimes 1) \circ H^*(\varepsilon \otimes 1)^{-1}$ .

The usual proof [20, p. 26] that the Eilenberg-Moore isomorphism  $EM$  is an isomorphism of algebras with respect to the cup product gives the following commutative square

$$\begin{array}{ccc} H^*(LM)^{\otimes 2} & \xleftarrow[\substack{EM \otimes 2 \\ \times \downarrow \cong}]{} & \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I))^{\otimes 2}_{\Delta^*, p^*} \\ & & \cong \downarrow \tilde{\tau} \\ H^*(LM \times LM) & \xleftarrow[\cong]{EM_3} & \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^I \times M^I))_{\Delta^{2*}, p^{2*}}. \end{array}$$

This square is the top square in [33, p. 255].

Consider the commutative diagram of spaces where the three composites of the vertical morphisms are the diagonal maps

$$\begin{array}{ccccccc} M & \xlongequal{\quad} & M & \xrightarrow{\Delta} & M \times M & & \\ \sigma \downarrow \simeq & & \sigma' \downarrow \simeq & & \sigma \times \sigma \downarrow \simeq & & \\ M^I & \xleftarrow{c} & M^I \times_M M^I & \xrightarrow{\bar{q}} & M^I \times M^I & & \\ p \downarrow & & u \downarrow & & p \times p \downarrow & & \\ M \times M & \xleftarrow[p_{13}]{} & M^3 & \xrightarrow[w]{} & M^4 & & \end{array}$$

Using the homotopy equivalence  $\sigma$ ,  $\sigma'$  and  $\sigma^2$ , we have the result.  $\square$

We decompose the maps, which induce the loop coproduct, with pull-back diagrams. Let  $l : LM \rightarrow M \times M$  be a map defined by  $l(\gamma) = (\gamma(0), \gamma(\frac{1}{2}))$ . We define a map  $\varphi : LM \rightarrow LM$  by  $\varphi(\gamma)(t) = \gamma(2t)$  for  $0 \leq t \leq \frac{1}{2}$  and  $\varphi(\gamma)(t) = \gamma(1)$  for  $\frac{1}{2} \leq t \leq 1$ . Then  $\varphi$  is homotopic to the identity map and fits into the commutative diagram

$$(3.5) \quad \begin{array}{ccccc} LM & \xrightarrow{j} & M^I & & \\ \varphi \swarrow \simeq \quad \downarrow ev_0 & & \downarrow ev_0, ev_1 = p & & \\ LM & \xrightarrow{l} & M^I \times M^I & \xrightarrow{\beta} & M \times M \\ \downarrow & \Delta \searrow & \downarrow p \times p & & \downarrow \alpha \\ M \times M & \xrightarrow{\gamma'} & M^4 & \xleftarrow{\alpha} & M \times M \end{array}$$

Here the maps  $\alpha : M^2 \rightarrow M^4$ ,  $\beta : M^I \times M^I \rightarrow M \times M$  and  $\gamma' : M^2 \rightarrow M^4$  are defined by  $\alpha(x, y) = (x, y, y, y)$ ,  $\beta(r) = (r, c_{r(1)})$  with the constant loop  $c_{r(1)}$  at  $r(1)$  and  $\gamma'(x, y) = (x, y, y, x)$ , respectively. We consider moreover the two pull-back squares

$$(3.6) \quad \begin{array}{ccccc} LM \times_M LM & \xrightarrow{Comp} & LM & \longrightarrow & M^I \times M^I \\ \downarrow & & l \downarrow & & \downarrow p \times p \\ M & \xrightarrow[\Delta]{} & M \times M & \xrightarrow[\gamma']{} & M^4 \end{array}$$

and the commutative cube

$$(3.7) \quad \begin{array}{ccccc} & LM \times_M LM & \xrightarrow{i \circ q} & M^I \times M^I & \\ q \swarrow & \downarrow i & & \searrow p^2 & \\ LM \times LM & & M^I \times M^I & & \\ \downarrow & \downarrow (\Delta \times \Delta) \circ \Delta & \downarrow p^2 & & \\ M \times M & \xrightarrow{\Delta} & M^4 & & \\ \Delta \times \Delta & & \searrow & & \end{array}$$

in which front and back squares are also pull-back diagrams.

*Proof of Theorem 2.5.* We see that the diagrams (3.5), (3.6) and (3.7) give rise to a commutative diagram

$$\begin{array}{ccccc} H^*(LM \times LM) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*((M^I)^2))_{\Delta^{2*}, p^{2*}} & & \\ q^* \downarrow & & \downarrow \text{Tor}_1(\Delta^*, 1) & & \\ H^*(LM \times_M LM) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^I \times M^I))_{(wv)^*, p^{2*}} & & \\ \text{Comp}^! \downarrow & & \downarrow \text{Tor}_1(\Delta^!, 1) & & \\ H^{*+m}(LM) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M^4)}^{*+m}(C^*(M^2), C^*(M^I \times M^I))_{\gamma'^*, p^{2*}} & & \\ \varphi^* = id \downarrow & & \downarrow \text{Tor}_{\alpha^*}(\Delta^*, \beta^*) \cong & & \\ H^{*+m}(LM) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M^2)}^{*+m}(C^*(M), C^*(M^I))_{\Delta^*, p^*}. & & \end{array}$$

In fact, the diagrams (3.5) and (3.7) give morphisms of pull-backs from the back face to the front face. Therefore the naturality of the Eilenberg-Moore map yields that the top and the bottom squares are commutative.

Using the diagram (3.6), the same argument as in the proof of Theorem 2.3 enables us to conclude that the middle square is commutative.

Since the following diagram of spaces

$$\begin{array}{ccccc} M & \xrightarrow{\Delta} & M \times M & & \\ \sigma \downarrow \simeq & & \sigma \times \sigma \downarrow \simeq & & \\ M^I & \xrightarrow{\beta} & M^I \times M^I & & \\ p \downarrow & & p \times p \downarrow & & \\ M \times M & \xrightarrow{\alpha} & M^4 & & \end{array}$$

is commutative, the theorem follows.  $\square$

By considering the free loop fibration  $\Omega M \xrightarrow{\tilde{\eta}} LM \xrightarrow{ev_0} M$ , we define for Gorenstein space (see Example 11.2) an intersection morphism  $H(\tilde{\eta}_! : H_{*+m}(LM) \rightarrow H_*(\Omega M)$

generalizing the one defined by Chas and Sullivan [4]. Using the following commutative cube

$$\begin{array}{ccccc}
 & LM & \longrightarrow & M^I & \\
 \tilde{\eta} \nearrow & \downarrow ev_0 & & \nearrow \widetilde{\eta \times 1} & \\
 \Omega M & \xrightarrow{\quad} & PM & \xrightarrow{\quad} & M \times M \\
 \downarrow & \downarrow & \downarrow ev_1 & \downarrow p & \\
 M & \xrightarrow{\Delta} & M \times M & & \\
 \downarrow \eta & \nearrow & \nearrow \widetilde{\eta \times 1} & & \\
 * \times M & \longrightarrow & * \times M & &
 \end{array}$$

where all the faces are pull-backs, we obtain similarly the following theorem.

**Theorem 3.1.** *Let  $M$  be a simply-connected Gorenstein with generator  $\omega_M$  in  $\text{Ext}_{C^*(M)}^m(\mathbb{K}, C^*(M))$ . Then the dual of the intersection morphism  $H(\tilde{\eta}^!)$  is given by the commutative diagram*

$$\begin{array}{ccccc}
 H^*(\Omega M) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M)}^*(\mathbb{K}, C^*(PM))_{\eta*, ev_1*} & \xrightarrow[\cong]{\text{Tor}_1^*(1, \eta^*)} & \text{Tor}_{C^*(M)}^*(\mathbb{K}, \mathbb{K})_{\eta*, \eta^*} \\
 \downarrow H(\tilde{\eta}^!) & \swarrow \cong & \uparrow \cong \text{Tor}_{(\eta \times 1)^*}^*(1, (\widetilde{\eta \times 1})^*) & & \uparrow \cong \text{Tor}_{(\eta \times 1)^*}^*(1, \eta^*) \\
 & EM & & & \\
 & \text{Tor}_{C^*(M^2)}^*(\mathbb{K}, C^*(M^I))_{\eta*, p^*} & \xrightarrow[\cong]{\text{Tor}_1^*(1, \sigma^*)} & \text{Tor}_{C^*(M^2)}^*(\mathbb{K}, C^*M)_{\eta*, \Delta^*} \\
 & \downarrow \text{Tor}_1^*(\omega_M, 1) & & & \downarrow \text{Tor}_1^*(\omega_M, 1) \\
 H^{*+m}(LM) & \xleftarrow[\cong]{EM} & \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I))_{\Delta*, p^*} & \xrightarrow[\cong]{\text{Tor}_1^*(1, \sigma^*)} & \text{Tor}_{C^*(M^2)}^*(C^*M, C^*M)_{\Delta*, \Delta^*}.
 \end{array}$$

Let  $\widehat{\mathcal{F}}$  be the pull-back diagram in the front of (3.1). Let  $\widetilde{\mathcal{F}}$  denote the pull-back diagram obtained by combining the front and the right hand-side squares in (3.1). Then a map inducing the isomorphism  $\text{Tor}_{w^*}(1, \widetilde{q}^*)$  gives rise to a morphism  $\{f_r\} : \{\widehat{E}_r, \widehat{d}_r\} \rightarrow \{\widetilde{E}_r, \widetilde{d}_r\}$  of spectral sequences, where  $\{\widehat{E}_r, \widehat{d}_r\}$  and  $\{\widetilde{E}_r, \widetilde{d}_r\}$  are the Eilenberg-Moore spectral sequences associated with the fibre squares  $\widehat{\mathcal{F}}$  and  $\widetilde{\mathcal{F}}$ , respectively. In order to prove Theorem 2.8, we need the following lemma.

**Lemma 3.2.** *The map  $f_2$  is an isomorphism.*

*Proof.* We identify  $f_2$  with the map

$$\text{Tor}_{H^*(w)}(1, H^*(\Delta)) : \text{Tor}_{H^*(M^4)}(H^*M, H^*(M \times M)) \rightarrow \text{Tor}_{H^*(M^3)}(H^*M, H^*M).$$

up to isomorphism between the  $E_2$ -term and the torsion product. Thus, in order to obtain the result, it suffices to apply part (1) of Lemma 10.3 for the algebra  $H^*(M)$  and the module  $H^*(M)$ .  $\square$

We are now ready to give the EMSS (co)multiplicative structures.

*Proof of Theorem 2.8.* Gugenheim and May [20, p. 26] have shown that the map  $\widetilde{\top}$  induces a morphism of spectral sequences from  $E_r \otimes E_r$  to the Eilenberg-Moore spectral sequence converging to  $H^*(LM \times LM)$ . In fact,  $\widetilde{\top}$  induces an isomorphism of spectral sequences. All the other maps between torsion products in Theorems 2.3 and 2.5 preserve the filtrations. Thus in view of Lemma 3.2, we have Theorem 2.8.

In fact, the shriek map  $\Delta^!$  is in  $\text{Ext}_{C^*(M^2)}^m(C^*(M), C^*(M^2))$ . Then we have  $d\Delta^! = (-1)^m \Delta^! d$ . Let  $\{\widehat{E}_r^{*,*}, \widehat{d}_r\}$  and  $\{\widetilde{E}_r^{*,*}, \widetilde{d}_r\}$  be the EMSS's converging to  $\text{Tor}_{C^*(M^4)}^*(C^*(M), C^*((M^I)^2))$  and  $\text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*((M^I)^2))$ , respectively. Let  $\{f_r\} : \{\widehat{E}_r^{*,*}, \widehat{d}_r\} \rightarrow \{\widetilde{E}_r^{*,*}, \widetilde{d}_r\}$  be the morphism of spectral sequences which

gives rise to  $\text{Tor}_1(\Delta^!, 1)$ . Recall the map  $\Delta^! \otimes 1 : \mathbb{B} \otimes_{C^*(M^4)} \mathbb{B}' \rightarrow C^*(M^2) \otimes_{C^*(M^4)} \mathbb{B}'$  in the proof of Theorem 2.3. It follows that, for any  $b \otimes b' \in \mathbb{B} \otimes_{C^*(M^4)} \mathbb{B}'$ ,

$$\begin{aligned} (\Delta^! \otimes 1)d(b \otimes b') &= \Delta^!(db \otimes b' + (-1)^{\deg b} b \otimes db') \\ &= \Delta^!db \otimes b' + (-1)^{\deg b} \Delta^!b \otimes db' \\ &= (-1)^m d\Delta^!b \otimes b' + (-1)^{\deg b} \Delta^!b \otimes db'. \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} d(\Delta^! \otimes 1)(b \otimes b') &= d\Delta^!(b \otimes b') \\ &= d\Delta^!b \otimes b' + (-1)^{\deg b+m} \Delta^!b \otimes db' \end{aligned}$$

and hence  $(\Delta^! \otimes 1)d = (-1)^m d(\Delta^! \otimes 1)$ . This implies that  $f_r \widehat{d}_r = (-1)^m \widetilde{d}_r f_r$ . The fact yields the compatibility of the multiplication with the differential of the spectral sequence.

The same argument does work well to show the compatibility of the comultiplication with the differential of the EMSS.  $\square$

*Proof of Corollary 2.9.* Since  $H^*(\Delta^!)$  is  $H^*(M^2)$ -linear, it follows that  $H^*(\Delta^!) \circ H^*(\Delta)(x) = H^*(\Delta^!)(1) \cup x$ . If  $d < 0$  then  $H^*(\Delta^!)(1) = 0$ .

If  $d = 0$  then  $H^*(\Delta^!)(1) = \lambda 1$  where  $\lambda \in \mathbb{K}$  and so the composite  $H^*(\Delta^!) \circ H^*(\Delta)$  is the multiplication by the scalar  $\lambda$ . Let  $m$  be an non-trivial element of positive degree in  $H^*(M)$ . Then we see that  $0 = H^*(\Delta^!) \circ H^*(\Delta)(m \otimes 1 - 1 \otimes m) = \lambda(m \otimes 1 - 1 \otimes m)$ . Therefore  $\lambda = 0$ .

So in both cases, we have proved that  $H^*(\Delta^!) \circ H^*(\Delta) = 0$ . Since  $H^*(\Delta)$  is surjective,  $H^*(\Delta^!) : H^*(M) \rightarrow H^{*+d}(M^2)$  is trivial. In particular, the induced maps  $\text{Tor}_{H^*(M^4)}^*(H^*(\Delta^!), H^*(M^I \times_M M^I))$  and  $\text{Tor}_{H^*(M^4)}^*(H^*(\Delta^!), H^*((M^I)^2))$  are trivial. Then it follows from Theorems 2.3 and 2.5 that both the comultiplication and the multiplication on the  $E_2$ -term of the EMSS, which correspond to the duals to loop product and loop coproduct on  $H^*(LM)$  are null. Therefore,  $E_\infty^{*,*} \cong \text{Gr}H^*(LM)$  is equipped with a trivial coproduct and a trivial product. Then the conclusion follows.  $\square$

*Remark 3.3.* It follows from Corollary 2.9 that under the hypothesis of Corollary 2.9 the two composites

$$H^*(M) \otimes H^*(M) \xrightarrow{p^* \otimes p^*} H^*(LM) \otimes H^*(LM) \xrightarrow{Dlcop} H^*(LM)$$

and

$$H_*(LM) \otimes H_*(LM) \xrightarrow{\text{Loop product}} H_*(LM) \xrightarrow{p_*} H_*(M)$$

are trivial. This can also be proved directly since we have the commuting diagram

$$\begin{array}{ccc} H^*(LM \times_M LM) & \xrightarrow{H(Comp^!)} & H^*(LM \times LM) \\ \uparrow & & \uparrow (p \times p)^* \\ H^*(M) & \xrightarrow{H(\Delta^!)} & H^*(M \times M) \end{array}$$

and since  $p_* : H_*(LM) \rightarrow H_*(M)$  is a morphism of graded algebras with respect to the loop product and to the intersection product  $H(\Delta^!)$ . As we saw in the proof of Corollary 2.9, under the hypothesis of Corollary 2.9,  $H^*(\Delta^!)$  and its dual  $H_*(\Delta^!)$  are trivial.

#### 4. PROOF OF THEOREM 2.13

The following Lemma is interesting on his own.

**Lemma 4.1.** *Let  $M$  be an oriented simply-connected Poincaré duality space of dimension  $m$ . Let  $M \rightarrow B$  be a fibration. Denote by  $M \times_B M$  the pull-back over  $B$ . Then for all  $p \in \mathbb{Z}$ ,  $\mathrm{Ext}_{C^*(B)}^{-p}(C^*(M), C^*(M))$  is isomorphic to  $H_{p+m}(M \times_B M)$  as a vector space.*

*Remark 4.2.* A particular case of Lemma 4.1 is the isomorphism of graded vector spaces  $\mathrm{Ext}_{C^*(M \times M)}^{-p}(C^*(M), C^*(M)) \cong H_{p+m}(LM)$  underlying the isomorphism of algebras given in Theorem 2.14. Note yet that in the proof of Lemma 4.1, we consider right  $C^*(B)$ -modules and that in the proof of Theorem 2.14, we need left  $C^*(M^2)$ -modules; see Section 10.

*Remark 4.3.* Let  $F$  be the homotopy fibre of  $M \rightarrow B$ . In [23, Theorem B], Klein shows in term of spectra that  $\mathrm{Ext}_{C_*(\Omega B)}^{-p}(C_*(F), C_*(F)) \cong H_{p+m}(M \times_B M)$  and so, using the Yoneda product [23, Theorem A],  $H_{*+m}(M \times_B M)$  is a graded algebra.

The isomorphism above and that in Lemma 4.1 make us aware of *duality* on the extension functors of (co)chain complexes of spaces. As mentioned in the Introduction, this is one of topics in [27].

*Proof of Lemma 4.1.* The Eilenberg-Moore map gives an isomorphism

$$H_{p+m}(M \times_B M) \cong \mathrm{Ext}_{C^*(B)}^{-p-m}(C^*(M), C_*(M)).$$

The cap with a representative  $\eta$  of the fundamental class  $[M] \in H_m(M)$  gives a quasi-isomorphism of right- $C^*(M)$ -modules of upper degree  $-m$ ,

$$\mathrm{cap}_M : C^*(M) \xrightarrow{\sim} C_{m-*}(M), x \mapsto \eta \cap x.$$

Therefore, we have an isomorphism

$$\mathrm{Ext}_{C^*(B)}^*(C^*(M), \mathrm{cap}_M) : \mathrm{Ext}_{C^*(B)}^{-p}(C^*(M), C^*(M)) \rightarrow \mathrm{Ext}_{C^*(B)}^{-p-m}(C^*(M), C_*(M))$$

This completes the proof.  $\square$

*Proof of Theorem 2.13.* Theorems 2.3 and 2.5 allow us to describe part of the composite

$$H^*(LM \times_M LM) \xrightarrow{H(q^!)} H^{*+m}(LM \times LM) \xrightarrow{Dlcop} H^{*+m}(LM \times LM)$$

in terms of the following composite of appropriate maps between torsion functors

$$\begin{array}{c}
 \mathrm{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^2))_{(wv)^*, \Delta^{2*}} \\
 \downarrow \mathrm{Tor}_1(\Delta^!, 1) \\
 \mathrm{Tor}_{C^*(M^4)}^{*+m}(C^*(M^2), C^*(M^2))_{\Delta^{2*}, \Delta^{2*}}. \\
 \downarrow \mathrm{Tor}_1(\Delta^*, 1) \\
 \mathrm{Tor}_{C^*(M^4)}^{*+m}(C^*(M), C^*(M^2))_{(wv)^*, \Delta^{2*}} \\
 \downarrow \mathrm{Tor}_1(\Delta^!, 1) \\
 \mathrm{Tor}_{C^*(M^4)}^{*+2m}(C^*(M^2), C^*(M^2))_{\gamma'^*, \Delta^{2*}} \\
 \downarrow \mathrm{Tor}_1(\Delta^*, 1) \quad \searrow \mathrm{Tor}_{\alpha^*}(\Delta^*, \Delta^*) \cong \\
 \mathrm{Tor}_{C^*(M^4)}^{*+2m}(C^*(M^2), C^*(M^2))_{\Delta^*, \alpha^*, \Delta^{2*}} \xrightarrow{\cong} \mathrm{Tor}_{C^*(M^2)}^{*+2m}(C^*M, C^*M)_{\Delta^*, \Delta^*}.
 \end{array}$$

By virtue of Lemma 4.1, we see that  $\text{Ext}_{C^*(M^4)}^{2m}(C^*(M), C^*(M))_{(wv)^*, \Delta^* \alpha^*}$  is isomorphic to  $H_{-m}(M^{S^1 \vee S^1 \vee S^1}) = \{0\}$ . Then the composite  $C^*(M) \xrightarrow{\Delta^!} C^*(M \times M) \xrightarrow{\Delta^*} C^*(M) \xrightarrow{\Delta^!} C^*(M \times M) \xrightarrow{\Delta^*} C^*(M)$  is null in  $D(\text{Mod-}C^*(M^4))$ . Therefore the composite  $Dlcop \circ H(q^!)$  is trivial and hence  $Dlcop \circ Dlp := Dlcop \circ H(q^!) \circ comp^*$  is also trivial.  $\square$

*Remark 4.4.* Instead of using Lemma 4.1, one can show that

$$\text{Ext}_{C^*(M^4)}^{2m}(C^*(M), C^*(M))_{(wv)^*, \Delta^* \alpha^*} = \{0\}$$

as follow: Consider the cohomological Eilenberg-Moore spectral sequence with

$$\mathbb{E}_2^{p,*} \cong \text{Ext}_{H^*(M^4)}^p(H^*(M), H^*(M))$$

converging to  $\text{Ext}_{C^*(M^4)}^*(C^*(M), C^*(M))_{(wv)^*, \Delta^* \alpha^*}$ . Then we see that

$$\mathbb{E}_1^{p,*} = \text{Hom}(H^*(M) \otimes H^+(M^4)^{\otimes p}, H^*(M)).$$

Therefore, since  $M^4$  is simply-connected and  $H^{>m}(M) = \{0\}$ ,  $\mathbb{E}_r^{p,q} = \{0\}$  if  $q > m - 2p$  (Compare with Remark 5.5). Therefore  $\text{Ext}_{C^*(M^4)}^{p+q}(C^*(M), C^*(M))_{(wv)^*, \Delta^* \alpha^*} = \{0\}$  if  $p + q > m$ .

*Remark 4.5.* Let  $M$  be a Gorenstein space of dimension  $m$ . The proof of Theorem 2.12 shows that if the composite

$$\Delta^* \circ \Delta^! \circ \Delta^* \circ \Delta^! \in \text{Ext}_{C^*(M^4)}^{2m}(C^*(M), C^*(M))_{(wv)^*, \Delta^* \alpha^*}$$

is the zero element. Then  $Dlcop \circ Dlp$  trivial.

*Remark 4.6.* In the proof of Theorem 2.13, it is important to work in the derived category of  $C^*(M^4)$ -modules: Suppose that  $M$  is the classifying space of a connected Lie group of dimension  $-m$ . Then since  $m$  is negative, the composite  $\Delta^! \circ \Delta^*$  is null. In fact  $\Delta^! \circ \Delta^* \in \text{Ext}_{C^*(M^2)}^m(C^*(M^2), C^*(M^2))_{1^*, 1^*} \cong H^m(M^2) = \{0\}$ . But in general,  $Dlcop$  is not trivial; see [15, Theorem D] and [26]. Therefore the composite  $\Delta^* \circ \Delta^! \circ \Delta^* \in \text{Ext}_{C^*(M^4)}^m(C^*(M^2), C^*(M))_{\Delta^{2*}, \Delta^* \alpha^*}$  is also not trivial.

## 5. PROOF OF THEOREM 2.11

We prove the following particular version of Theorem 2.11.

**Theorem 5.1.** *Let  $M$  be a simply-connected Poincaré duality space. Then the Eilenberg-Moore spectral sequence is a right-half plane cohomological spectral sequence  $\{\mathbb{E}_r^{*,*}, d_r\}$  converging to the Chas-Sullivan loop homology  $\mathbb{H}_*(LM)$  as an algebra; that is, there exists a decreasing filtration  $\{F^p \mathbb{H}_*(LM)\}_{p \geq 0}$  of  $\mathbb{H}_*(LM)$  such that  $\mathbb{E}_\infty^{*,*} \cong Gr^{*,*} \mathbb{H}_*(LM)$  as a bigraded algebra, where*

$$Gr^{p,q} \mathbb{H}_*(LM) = F^p \mathbb{H}_{-(p+q)}(LM) / F^{p+1} \mathbb{H}_{-(p+q)}(LM)$$

and

$$\mathbb{E}_2^{*,*} \cong HH^{*,*}(H^*(M); H^*(M))$$

as a bigraded algebra.

The proof of Theorem 2.11 proceeds verbatim as that of Theorem 5.1. Indeed we have a torsion functor description of the relative loop product as in Theorem 2.3; see the proof of Proposition 9.2 below. The details are left to the reader.

Let  $A$  be a Poincaré duality DGA. Then Theorem 2.3 defines a coproduct on the torsion product  $\text{Tor}_{A^{\otimes 2}}(A, A)$ . It turns out that the dual to the coproduct

coincides with the cup product on the Hochschild cohomology  $HH^*(A, A)$  under an appropriate isomorphism between  $\text{Tor}_{A \otimes 2}(A, A)$  and  $HH^*(A, A)$ . To state the result more precisely, we recall a Poincaré duality DGA.

**Definition 5.2.** An oriented differential Poincaré duality algebra over an arbitrary field  $\mathbb{K}$  of dimension  $d$  is a triple  $(A, d, \varepsilon)$  such that

- (1)  $(A, d)$  is a connected commutative differential graded algebra,
- (2)  $(A, \varepsilon)$  is an oriented Poincaré duality algebra; that is,  $\varepsilon : A^d \rightarrow \mathbb{K}$  such that the induced bilinear forms  $A^k \otimes A^{d-k} \rightarrow \mathbb{K}$ ,  $a \otimes b \mapsto \varepsilon(ab)$  are non-degenerate,
- (3)  $\varepsilon(dA) = 0$ .

We see that the map  $\theta_A : A \rightarrow A^\vee$  of degree  $-d$  defined by  $\theta_A(a)(b) = \varepsilon(ab)$  for  $a, b \in A$  is a right  $A$ -linear isomorphism which commutes with differential. Observe that  $A^d \cong (A^0)^\vee \cong \mathbb{K}$  and  $A^p = 0$  for  $p > d$ .

Denote by  $\mu : A \otimes A \rightarrow A$  the product of  $A$ . We define the map  $\Delta^! : A \rightarrow A \otimes A$  by the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Delta^!} & A \otimes A \\ \theta_A \downarrow & & \downarrow \theta_A \otimes \theta_A \\ A^\vee & \xrightarrow[\mu^\vee]{} & (A \otimes A)^\vee \xleftarrow[\cong]{} A^\vee \otimes A^\vee. \end{array}$$

By definition,  $\Delta^!$  is a right  $A^{\otimes 2}$ -linear map of degree  $d$  which commutes with the differential.

Let  $\xi : \mathbb{B} = B(A, A, A) \rightarrow A$  be the bar resolution of  $A$ . Recall from [16] the coproduct  $\bar{c} : \mathbb{B} \rightarrow \mathbb{B} \otimes_A \mathbb{B}$  of  $\mathbb{B}$  defined by

$$\bar{c}(a[a_1|a_2|\cdots|a_n]b) = \sum_{i=0}^n a[a_1|\cdots|a_i]1 \otimes 1[a_{i+1}|\cdots|a_n]b \quad (n \geq 0).$$

We denote by  $\bar{q} : \mathbb{B} \otimes \mathbb{B} \rightarrow \mathbb{B} \otimes_A \mathbb{B}$  the quotient map. Moreover we define maps, which are needed below, as follows.

$$\begin{aligned} \bar{p} : A \otimes A &\longrightarrow \mathbb{B}, \quad \bar{p}(a \otimes b) = a[ ]b, \\ \bar{u} : A^{\otimes 3} &\longrightarrow \mathbb{B} \otimes_A \mathbb{B}, \quad \bar{u}(a_1 \otimes a_2 \otimes a_3) = a_1[ ]a_2 \otimes 1[ ]a_3, \\ \bar{v} := \mu(\mu \otimes 1) : A^{\otimes 3} &\longrightarrow A, \\ \bar{w} := 1 \otimes \mu \otimes 1 : A^{\otimes 4} &\longrightarrow A^{\otimes 3}, \\ \overline{p_{13}} : A^{\otimes 2} &\longrightarrow A^{\otimes 3}, \quad \overline{p_{13}}(a \otimes b) = a \otimes 1 \otimes b. \end{aligned}$$

The diagram in Theorems 2.3 allows us to define the map  $Dlp_A : \text{Tor}_{A \otimes 2}^*(A, A) \rightarrow (\text{Tor}_{A \otimes 2}^*(A, A))^{\otimes 2}$  of degree  $+d$  by the composite

$$\begin{array}{ccccc} Dlp_A : \text{Tor}_{A \otimes 2}^*(A, \mathbb{B})_{\mu, \bar{p}} & \xrightarrow{\text{Tor}_{\overline{p_{13}}(1, \bar{c})}^{-1}} & \text{Tor}_{A \otimes 3}^*(A, \mathbb{B} \otimes_A \mathbb{B})_{\bar{v}, \bar{u}} & \xrightarrow{\text{Tor}_{\bar{w}}(1, \bar{q})^{-1}} & \text{Tor}_{A \otimes 4}^*(A, \mathbb{B} \otimes \mathbb{B})_{\bar{w}\bar{v}, \bar{p} \otimes 2} \\ & & \searrow \text{Tor}_1(\Delta^!, 1) & & \\ & & \text{Tor}_{A \otimes 4}^{*+d}(A^{\otimes 2}, \mathbb{B} \otimes \mathbb{B})_{\mu \otimes 2, \bar{p} \otimes 2} & \xrightarrow[\phi^{-1}]{} & (\text{Tor}_{A \otimes 2}^*(A, \mathbb{B})_{\mu, \bar{p}})^{*+d}, \end{array}$$

where  $\phi : \text{Tor}_{A \otimes 2}^*(A, \mathbb{B})_{\mu, \bar{p}}^{\otimes 2} \rightarrow \text{Tor}_{A \otimes 4}^*(A^{\otimes 2}, \mathbb{B} \otimes \mathbb{B})_{\mu \otimes 2, \bar{p} \otimes 2}$  is an isomorphism defined by  $\phi((a_1 \otimes x_1) \otimes (a_2 \otimes x_2)) = (-1)^{|x_1||a_2|}(a_1 \otimes a_2) \otimes (x_1 \otimes x_2)$ , for  $a_i \otimes x_i \in A \otimes_{A \otimes 2} \mathbb{B}$ .

We relate the map  $Dlp_A$  to the cup product on the Hochschild cohomology  $HH^*(A; A)$ . Let  $\zeta : HH^*(A, A) \rightarrow \text{Tor}_{A \otimes 2}^*(A, \mathbb{B})^\vee$  be the map of degree  $-d$  defined by the composite

$$\begin{array}{ccccc} HH^*(A, A) = H^*(\text{Hom}_{A \otimes 2}(\mathbb{B}, A)) & \xrightarrow{\theta_{A^*}} & H^{*-d}(\text{Hom}_{A \otimes 2}(\mathbb{B}, A^\vee)) \\ & \swarrow \iota_* & & & \\ H^{*-d}(\text{Hom}_\mathbb{K}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K})) & \xrightarrow[\cong]{} & \text{Hom}_\mathbb{K}(H^{d-*}(A \otimes_{A \otimes 2} \mathbb{B}), \mathbb{K}) = (\text{Tor}_{A \otimes 2}(A, A)^\vee)^{*-d}. \end{array}$$

Here  $\iota_*$  denotes the adjunction map induced by the isomorphism of complexes

$$\iota : \text{Hom}_{A \otimes 2}(\mathbb{B}, A^\vee) \longrightarrow \text{Hom}_\mathbb{K}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K})$$

which is defined by  $\iota(\varphi)(a \otimes x) = (-1)^{|x||a|}\varphi(x)(a)$  for  $a \in A$  and  $x \in \mathbb{B}$ . We observe that  $\zeta$  is an isomorphism of vector spaces.

Let  $\cup : HH^*(A, A) \otimes HH^*(A, A) \rightarrow HH^*(A, A)$  be the cup product of  $HH^*(A, A)$ ; that is, for  $\varphi_i \in HH^*(A, A)$ , the product is defined by

$$(5.1) \quad (\varphi_1 \cup \varphi_2)(a[a_1 | \cdots | a_m]a') = (\mu \circ (\varphi_1 \otimes \varphi_2) \circ \bar{c})(a[a_1 | \cdots | a_m]a').$$

We then have the following result.

**Theorem 5.3.** *The diagram*

$$\begin{array}{ccc} ((\text{Tor}_{A \otimes 2}(A, A)^\vee)^{\otimes 2})^{*-2d} & \xleftarrow[\cong]{\zeta \otimes \zeta} & HH^*(A, A)^{\otimes 2} \\ \cong \downarrow & & \downarrow \cup \\ ((\text{Tor}_{A \otimes 2}(A, A)^{\otimes 2})^\vee)^{*-2d} & & \\ (\text{Dlp}_A)^\vee \downarrow & & \downarrow \\ (\text{Tor}_{A \otimes 2}(A, A)^\vee)^{*-d} & \xleftarrow[\cong]{\zeta} & HH^*(A, A) \end{array}$$

is commutative.

To prove Theorem 5.3, we shall use an interesting decomposition of the cup product of the Hochschild cohomology of a commutative (possibly differential graded) algebra, which is described in Lemma 10.3 below.

*Proof of Theorem 5.3.* We consider the following two diagrams

$$\begin{array}{ccccccc} H^*(\text{Hom}_\mathbb{K}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K}))^{\otimes 2} & \xleftarrow{\iota_* \otimes \iota_*} & H^*(\text{Hom}_{A \otimes 2}(\mathbb{B}, A^\vee))^{\otimes 2} & \xleftarrow{\theta_{A^*} \otimes \theta_{A^*}} & HH^*(A, A)^{\otimes 2} & & \\ \downarrow \otimes & & \downarrow \otimes & & \downarrow \otimes & & \\ H^*(\text{Hom}_{A \otimes 4}(\mathbb{B}^{\otimes 2}, (A^\vee)^{\otimes 2}))^{(\theta_A \otimes \theta_A)_*} & \xleftarrow[\cong]{} & H^*(\text{Hom}_{A \otimes 4}(\mathbb{B}^{\otimes 2}, A^{\otimes 2}))^{(\theta_A \otimes \theta_A)_*} & & & & \\ \downarrow \text{Hom}_1(1, \mu) & & \downarrow \text{Hom}_1(1, \mu) & & \downarrow \text{Hom}_1(1, \mu) & & \\ H^*(\text{Hom}_\mathbb{K}(A^{\otimes 2} \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2}, \mathbb{K})) & \xleftarrow{\iota_*} & H^*(\text{Hom}_{A \otimes 4}(\mathbb{B}^{\otimes 2}, (A^{\otimes 2})^\vee)) & & & & \\ \downarrow \text{Hom}_1(\Delta^! \otimes 1, 1)_* & & \downarrow \text{Hom}_1(1, (\Delta^!)^\vee) & & & & \\ H^*(\text{Hom}_\mathbb{K}(A \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2}, \mathbb{K})) & \xleftarrow{\iota_*} & H^*(\text{Hom}_{A \otimes 4}(\mathbb{B}^{\otimes 2}, A^\vee)) & \xleftarrow{\theta_{A^*}} & H^*(\text{Hom}_{A \otimes 4}(\mathbb{B}^{\otimes 2}, A)) & & \\ \downarrow \text{Hom}_1(1 \otimes q, 1)_*^{-1} & & \downarrow \text{Hom}_1(q, 1)_*^{-1} & & \downarrow \text{Hom}_1(q, 1)_*^{-1} & & \\ H^*(\text{Hom}_\mathbb{K}(A \otimes_{A \otimes 3} (\mathbb{B} \otimes_A \mathbb{B}), \mathbb{K})) & \xleftarrow{\iota_*} & H^*(\text{Hom}_{A \otimes 3}(\mathbb{B} \otimes_A \mathbb{B}, A^\vee)) & \xleftarrow{\theta_{A^*}} & H^*(\text{Hom}_{A \otimes 3}(\mathbb{B} \otimes_A \mathbb{B}, A)) & & \\ \downarrow \text{Hom}_1(1 \otimes \bar{c}, 1)_* & & \downarrow \text{Hom}_{\overline{P13}}(\bar{c}, 1) & & \downarrow \text{Hom}_{\overline{P13}}(\bar{c}, 1) & & \\ H^*(\text{Hom}_\mathbb{K}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K})) & \xleftarrow{\iota_*} & H^*(\text{Hom}_{A \otimes 2}(\mathbb{B}, A^\vee)) & \xleftarrow{\theta_{A^*}} & H^*(\text{Hom}_{A \otimes 2}(\mathbb{B}, A)) & & \end{array}$$

and

$$(5.3) \quad \begin{array}{ccccc} H^*(A \otimes_{A \otimes 2} \mathbb{B})^\vee \otimes H^*(A \otimes_{A \otimes 2} \mathbb{B})^\vee & \xleftarrow{\cong} & H^*(\text{Hom}_{\mathbb{K}}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K}))^{\otimes 2} & & \\ \downarrow \cong & & & & \downarrow \otimes \\ (H^*(A \otimes_{A \otimes 2} \mathbb{B}) \otimes H^*(A \otimes_{A \otimes 2} \mathbb{B}))^\vee & & & & \\ \otimes^\vee \uparrow \cong & & & & \\ H^*(A^{\otimes 2} \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2})^\vee & \xleftarrow{\cong} & H^*(\text{Hom}_{\mathbb{K}}(A^{\otimes 2} \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2}, \mathbb{K})) & & \\ (\Delta^! \otimes 1)_*^\vee \downarrow & & \text{Hom}_1(\Delta^! \otimes 1, 1)_* \downarrow & & \\ H^*(A \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2})^\vee & \xleftarrow{\cong} & H^*(\text{Hom}_{\mathbb{K}}(A \otimes_{A \otimes 4} \mathbb{B}^{\otimes 2}, \mathbb{K})) & & \\ ((1 \otimes q)_*^{-1})^\vee \downarrow & & \text{Hom}_1(1 \otimes q, 1)_*^{-1} \downarrow & & \\ H^*(A \otimes_{A \otimes 3} (\mathbb{B} \otimes_A \mathbb{B}))^\vee & \xleftarrow{\cong} & H^*(\text{Hom}_{\mathbb{K}}(A \otimes_{A \otimes 3} (\mathbb{B} \otimes_A \mathbb{B}), \mathbb{K})) & & \\ (1 \otimes \bar{c})_*^\vee \downarrow & & \text{Hom}_1(1 \otimes \bar{c}, 1) \downarrow & & \\ H^*(A \otimes_{A \otimes 2} \mathbb{B})^\vee & \xleftarrow{\cong} & H^*(\text{Hom}_{\mathbb{K}}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K})). & & \end{array}$$

$(Dlp_A)^\vee$

It is readily seen that all squares are commutative. In fact we see that the diagram  $(\star)$  in (5.1) is commutative by the definition of  $\Delta^!$ . Lemma 10.3 (4) enables us to conclude that the composite of the right hand side maps of the diagram (5.1)

$$\text{Hom}_{\overline{P_{13}}}(\bar{c}, 1)_* \text{Hom}_1(q, 1)^{-1} \text{Hom}_1(1, \mu) \otimes : HH^*(A, A)^{\otimes 2} \longrightarrow HH^*(A, A)$$

is the cup product of  $HH^*(A, A)$ . Therefore, by combining the diagrams (5.1) and (5.2), we have the result.  $\square$

*Remark 5.4.* Suppose that  $A$  is a Poincaré duality algebra of dimension  $d$  with trivial differential. Then the torsion functor  $\text{Tor}_{A \otimes 2}(A, A)$  and the Hochschild cohomology  $H^*(A, A)$  are bigraded modules. Under the bigrading, the Hochschild cohomology  $H^*(A, A)$  is a bigraded algebra with respect to the cup product.

Since the shriek map  $\Delta^!$  is of bidegree  $(0, d)$ , it follows that  $Dlp$  is a map of bidegree  $(0, d)$ . Moreover, we see that the isomorphism  $\zeta : \text{Tor}_{A \otimes 2}^{*, *}(A, A)^\vee \rightarrow HH^{*, *}(A, A)$  is of bidegree  $(0, -d)$ .

*Proof of Theorem 5.1.* Let  $M$  be a Poincaré duality space of dimension  $d$ . Let  $\{E_r^{*, *}, d_r\}$  denote the spectral sequence described in Theorem 2.8. Then we define a spectral sequence  $\{\mathbb{E}_r^{*, *}, d_r\}$  by

$$\mathbb{E}_r^{p, q+d} := (E_r^{*, *})^{p, q} = (E_r^{-p, -q})^\vee.$$

The decreasing filtration  $\{F^p H^*(LM)\}_{p \leq 0}$  of  $H^*(LM)$  induces the decreasing filtration  $\{F^p H_*(LM)\}_{p \geq 0}$  of  $H_*(LM)$  defined by  $F^p H_*(LM) = (H^*(LM)/F^{-p} H^*(LM))^\vee$ . By definition, the Chas-Sullivan loop homology  $\mathbb{H}_*(LM)$  is given by  $\mathbb{H}_{-(p+q)}(LM) = (H^*(LM)^\vee)^{p+q-d}$ . We observe that the product  $m$  on  $\mathbb{H}_*(LM)$  is defined by

$$m(a \otimes b) = (-1)^{d(|a|-d)} (Dlp)^\vee(a \otimes b)$$

for  $a \otimes b \in (H^*(LM)^\vee)^* \otimes (H^*(LM)^\vee)^*$ ; see Lemma 11.8. Then we see that

$$\mathbb{E}_\infty^{p, q} \cong F^p H_{p+q-d}(LM)/F^{p+1} H_{p+q-d}(LM) = F^p \mathbb{H}_{-(p+q)}(LM)/F^{p+1} \mathbb{H}_{-(p+q)}(LM).$$

The composite  $(\widetilde{Dlp})$  in Theorem 2.3 which gives rise to  $Dlp$  on  $H^*(LM)$  preserves the filtration of the EMSS  $\{E_r^{*, *}, d_r\}$ . As mentioned in the proof of Theorem 2.8, the map  $(\widetilde{Dlp})$  induces the morphism  $(Dlp)_r : E_r^{*, *} \rightarrow E_r^{*, *} \otimes E_r^{*, *}$  of spectral sequences of bidegree  $(0, d)$ . Define  $m_r : \mathbb{E}_r^{*, *} \otimes \mathbb{E}_r^{*, *} \rightarrow \mathbb{E}_r^{*, *}$  by

$$m_r(a \otimes b) = (-1)^{d(|a|+d)} ((Dlp)_r)^\vee(a \otimes b),$$

where  $|a| = p+q$  if  $a \in (E_r^{*,*})^{p,q}$ . Then a straightforward computation enables us to deduce that  $m_r \circ (d_r^\vee \otimes \pm 1 \otimes d_r^\vee) = d_r^\vee \circ m_r$  for any  $r$ . It turns out that  $\{\mathbb{E}_r, d_r^\vee\}$  is a spectral sequence converging to  $\mathbb{H}_{-*}(LM)$  as an algebra.

In order to show the latter half of Theorem 5.1, it suffices to prove that the map  $m_2$  is compatible with the cup products on  $HH^{*,*}(A, A)$  under an appropriate isomorphism.

Let  $A$  denote the cohomology  $H^*(M)$ . The map  $\zeta$  in Theorem 5.3 induces an isomorphism

$$\begin{aligned} \mathbb{E}_2^{p,q+d} &= (E_2^{-p,-q})^\vee \cong (\mathrm{Tor}_A^{-p,-q}(A, A))^\vee \\ &= (\mathrm{Tor}_A^{*,*}(A, A)^\vee)^{p,q} \xleftarrow[\cong]{\zeta} HH^{p,q+d}(A, A). \end{aligned}$$

We observe that the shriek map  $\Delta^! : H^*(M) \rightarrow H^*(M \times M)$  sends the dual  $[M]^\vee$  to the fundamental class of  $M$  to the dual  $[M \times M]^\vee$  to that of  $M \times M$  while the map  $\Delta^! : A \rightarrow A \otimes A$  defined after Definition 5.2 sends  $[M]^\vee$  to  $(-1)^d [M]^\vee \otimes [M]^\vee$ . This yields that the map  $(Dlp)_2$  on  $E_2^{*,*}$  coincides with  $Dlp_A$  up to multiplication by  $(-1)^d$ ; see the proof of [15, Lemma]. Then it follows from Theorem 5.3 that the map  $(-1)^d \zeta$  induces an isomorphism from  $HH^{*,*}(A, A)$  to  $\mathbb{E}_2^{*,*}$  of bigraded algebras. Observe that the isomorphism  $(-1)^d \zeta : HH^{*,*}(A, A) \rightarrow (\mathrm{Tor}_{A \otimes 2}(A, A)^\vee)^{*,*-d} \cong \mathbb{E}_2^{*,*}$  is of bidegree  $(0, 0)$ . We have the result.  $\square$

*Remark 5.5.* For the EMSS  $\{E_r^{*,*}, d_r\}$  described in Theorem 2.8, we see that  $E_r^{p,q} = 0$  if  $q < -2p$  since  $M$  is simply-connected. This implies that  $\mathbb{E}_r^{p,q} = 0$  if  $q > -2p+d$ .

## 6. ASSOCIATIVITY OF THE LOOP PRODUCT ON A POINCARÉ DUALITY SPACE

In this short section, by applying the same argument as in the proof of [46, Theorem 2.2], we shall prove the associativity of the loop products.

*Proof of Proposition 2.7.* We prove the proposition in the case where  $N = M$ . The same argument as in the proof permits us to conclude that the loop homology  $\mathbb{H}_*(L_N M)$  is associative with respect to the relative loop products.

Let  $M$  be a simply-connected Gorenstein space of dimension  $d$ . In order to prove the associativity of the dual to  $Dlp$ , we first consider the diagram

$$\begin{array}{ccccc} LM \times LM & \xleftarrow{\mathrm{Comp} \times 1} & (LM \times_M LM) \times LM & \xrightarrow{q \times 1} & LM \times LM \times LM \\ q \uparrow & & \uparrow 1 \times_M q & & \uparrow 1 \times q \\ LM \times_M LM & \xleftarrow{\mathrm{Comp} \times_M 1} & LM \times_M LM \times_M LM & \xrightarrow{q \times_M 1} & LM \times (LM \times_M LM) \\ \mathrm{Comp} \downarrow & & \downarrow 1 \times_M \mathrm{Comp} & & \downarrow 1 \times \mathrm{Comp} \\ LM & \xleftarrow[\mathrm{Comp}]{\mathrm{Comp}} & LM \times_M LM & \xrightarrow[q]{\quad} & LM \times LM, \end{array}$$

for which the lower left hand-side square is homotopy commutative and other three square are strictly commutative. Consider the corresponding diagram

$$\begin{array}{ccccc} H^*(LM \times LM) & \xrightarrow{(Comp \times 1)^*} & H^*((LM \times_M LM) \times LM) & \xrightarrow{\varepsilon' \alpha H(q^!) \otimes 1} & H^*(LM \times LM \times LM) \\ H(q^!) \uparrow & & \uparrow H((1 \times_M q)^!) & & \uparrow \varepsilon \alpha' 1 \otimes H(q^!) \\ H^*(LM \times_M LM) & \xrightarrow[(Comp \times_M 1)^*]{} & H^*(LM \times_M LM \times_M LM) & \xrightarrow{H((q \times_M 1)^!)} & H^*(LM \times (LM \times_M LM)) \\ Comp^* \uparrow & & (1 \times_M Comp)^* \uparrow & & \uparrow (1 \times Comp)^* \\ H^*(LM) & \xrightarrow[\mathrm{Comp}^*]{\mathrm{Comp}^*} & H^*(LM \times_M LM) & \xrightarrow[H(q^!)]{\quad} & H^*(LM \times LM). \end{array}$$

The lower left square commutes obviously. By Theorem 11.5, the upper left square and the lower right square are commutative. We now show that the upper right square commutes.

By Theorem 11.6, we see that  $H((q \times 1)!) = \alpha H(q^!) \otimes 1$  and  $H((1 \times q)!) = \alpha' 1 \otimes H(q^!)$  where  $\alpha$  and  $\alpha' \in \mathbb{K}^*$ . By virtue of [15, Theorem C], in  $D(\text{Mod-}C^*(LM^{\times 3}))$ ,

$$\varepsilon'(\Delta \times 1)! \circ \Delta^! = \varepsilon(1 \times \Delta)! \circ \Delta^!$$

where  $(\varepsilon, \varepsilon') \neq (0, 0) \in \mathbb{K} \times \mathbb{K}$ . Therefore the uniqueness of the shriek map implies that

$$\varepsilon'(q \times 1)! \circ (1 \times_M q)^! = \varepsilon(1 \times q)^! \circ (q \times_M 1)^!$$

in  $D(\text{Mod-}C^*(LM^{\times 3}))$ ; see [15, Theorem 13].

So finally, we have proved that

$$\varepsilon' \alpha(Dlp \otimes 1) \circ Dlp = \varepsilon \alpha'(1 \otimes Dlp) \circ Dlp.$$

Suppose that  $M$  is a Poincaré duality space of dimension  $d$ . By part (2) of Theorem 11.6,  $\alpha = 1$  and  $\alpha' = (-1)^d$ . Since  $\varepsilon(\omega_M \times \omega_M \times \omega_M) = \varepsilon H((\Delta \times 1)!) \circ H(\Delta^!(\omega_M)) = \varepsilon' H((1 \times \Delta)!) \circ H(\Delta^!(\omega_M)) = \varepsilon'(\omega_M \times \omega_M \times \omega_M)$ , we see that  $\varepsilon = \varepsilon'$ . Therefore  $(Dlp \otimes 1) \circ Dlp = (-1)^d (1 \otimes Dlp) \circ Dlp$ . Thus Lemma 11.8(i) below yields that the product  $m : \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \rightarrow \mathbb{H}_*(LM)$  is associative.

We prove that the loop product is graded commutative. Consider the commutative diagram

$$\begin{array}{ccccc} & LM \times_M LM & \xrightarrow{q} & LM \times LM & \\ T \swarrow & \downarrow q & & \searrow T & \\ LM \times_M LM & \xrightarrow{q} & LM \times LM & \xrightarrow{p \times p} & M \times M \\ \downarrow p & & \downarrow p \times p & & \downarrow \Delta \\ M & \xrightarrow{\Delta} & M \times M & \xrightarrow{T} & M \times M \end{array}$$

By Theorem 11.3 below,  $H(q^!) \circ T^* = \varepsilon T^* \circ H(q^!)$ . Since  $Comp \circ T$  is homotopic to  $Comp$ ,  $Dlp = \varepsilon T^* \circ Dlp$ . If  $M$  is a Poincaré duality space with orientation class  $\omega_M \in H^d(M)$  then  $T^*(\omega_M \otimes \omega_M) = (-1)^{d^2}(\omega_M \otimes \omega_M)$ . Therefore by part a) of Remark 11.4,  $\varepsilon = (-1)^d$ . By Lemma 11.8(ii) below, we see that the product  $m$  is graded commutative. The fact that  $\mathbb{H}_*(LM)$  is unital follows from Proposition 8.5. This completes the proof.  $\square$

*Remark 6.1.* The commutativity of the loop homology  $\mathbb{H}_*(L_N M)$  does not follow from the proof of Proposition 2.7. In general  $Comp \circ T$  is not homotopic to  $Comp$  in  $L_N M$ . As mentioned in the Introduction, the relative loop product is not necessarily commutative; see [42].

## 7. THE EMSS CALCULATIONS OF THE LOOP HOMOLOGY

By making use of the spectral sequence in Theorem 5.1 and the computation of the Hochschild cohomology of a graded commutative algebra, we may determine the loop cohomology of a Poincaré duality space.

The spectral sequence  $\{\mathbb{E}_r^{*,*}, d_r\}$  in Theorem 5.1 is constructed by dualizing the EMSS  $\{E_r, d_r\}$  converging to  $H^*(LM)$ . Therefore it is immediate that the EMSS  $\{E_r^{*,*}, d_r\}$  collapses at the  $E_2$ -term if and only if so does the EMSS  $\{\mathbb{E}_r^{*,*}, d_r\}$ . We thus establish the following theorem.

**Theorem 7.1.** *Let  $M$  be a simply-connected  $\mathbb{K}$ -Gorenstein space of positive dimension whose cohomology with coefficients in  $\mathbb{K}$  is generated by a single element of even degree. Then as an algebra,*

$$\mathbb{H}_{-*}(LM; \mathbb{K}) \cong HH^*(H^*(M; \mathbb{K}), H^*(M; \mathbb{K})).$$

*Remark 7.2.* Suppose that  $M$  is a simply-connected space whose cohomology with coefficients in a field  $\mathbb{K}$  is a finitely generated polynomial algebra, say  $H^*(M) \cong \mathbb{K}[x_1, \dots, x_n]$ . Let  $\mathbb{H}_*(LM)$  denote the shifted homology  $H_{*-d}(LM)$ , where  $d = -\sum_{i=1}^n (\deg x_i - 1)$ . Observe that  $M$  is a Gorenstein space of dimension  $d$  as seen in Remark 7.3 below. We have

$$\mathbb{H}_*(LM; \mathbb{K})^\vee \cong HH^*(H^*(M), H^*(M))$$

as a graded vector space. In fact, by using the Eilenberg-Moore spectral sequence converging to  $H^*(LM)$  with  $E_2^{*,*} \cong \text{Tor}_{H^*(M) \otimes H^*(M)}(H^*(M), H^*(M))$ , we see that

$$(\mathbb{H}_*(LM))^\vee = (H_{*-d}(LM))^\vee \cong H^{*-d}(LM) \cong (\mathbb{K}[x_1, \dots, x_n] \otimes \wedge(u_1, \dots, u_n))^{*-d}$$

as graded vector spaces, where  $\deg u_i = \deg x_i - 1$ . Moreover, it follows from [25, Theorem 1.1] that

$$HH^*(H^*(M), H^*(M)) \cong HH^*(C^*(M), C^*(M)) \cong \mathbb{K}[x_1, \dots, x_n] \otimes \wedge(u_1^*, \dots, u_n^*)$$

as algebras, where  $\deg u_i^* = -(\deg x_i - 1)$ . We define a map

$$\eta : HH^*(H^*(M), H^*(M)) \rightarrow H^{*-d}(LM)$$

by

$$\eta(x_{i_1} \cdots x_{i_s} u_{j_1}^* \cdots u_{j_t}^*) = x_{i_1} \cdots x_{i_s} u_1 \cdots \widehat{u_{j_1}} \cdots \widehat{u_{j_t}} \cdots u_n,$$

where  $\widehat{u_j}$  means deletion of the element  $u_j$  from the representation. Then it is readily seen that  $\eta$  is an isomorphism of graded vector spaces. See [38, Section 9] for such an isomorphism in more general setting.

*Remark 7.3.* Let  $M$  be the same space as in Remark 7.2. Then  $M$  is a  $\mathbb{K}$ -Gorenstein space of dimension  $d = -\sum_{i=1}^n (\deg x_i - 1)$ . In fact, since  $M$  is a  $\mathbb{K}$ -formal, it follows that

$$\begin{aligned} \text{Ext}_{C^*(M)}^*(\mathbb{K}, C^*(M)) &\cong \text{Ext}_{H^*(M)}^*(\mathbb{K}, H^*(M)) \\ &\cong (\otimes_{i=1}^n \text{Ext}_{\mathbb{K}[x_i]}^*(\mathbb{K}, \mathbb{K}[x_i]))^* = \begin{cases} \mathbb{K} & \text{if } * \neq d, \\ 0 & \text{if } * = d. \end{cases} \end{aligned}$$

The result [13, Theorem 6.10] allows us to obtain the first isomorphism. The proof of [10, (4.6)] gives us the second one.

We can choose a shriek map

$$\Delta' \in \text{Ext}_{C^*(M \times 2)}^d(C^*(M), C^*(M^{\times 2})) = \text{Ext}_{C^*(M \times 2)}^d(C^*(M^I), C^*(M^{\times 2})) = H^0(M)$$

so that  $H(\Delta')$  is the integration along the fibre of the fibration  $\Omega M \rightarrow M^I \rightarrow M^{\times 2}$ . Thus  $\mathbb{H}_*(LM)^\vee \cong H^{*-d}(LM)$  is endowed with  $Dlcop$  the dual to the loop coproduct defined in the Section 1. From Remark 7.2, one might expect that, as an algebra,  $\mathbb{H}_*(LM)^\vee$  is isomorphic to  $HH^*(H^*(M), H^*(M))$ . The consideration of such an isomorphism is one of main topics in [26]. We also mention that the dual to the loop product on  $\mathbb{H}_*(LM)^\vee$  is trivial; see [26] for more details.

As seen in Remark 7.3, a simply-connected space  $M$  is a  $\mathbb{K}$ -Gorenstein space of negative degree if the cohomology  $H^*(M; \mathbb{K})$  is a polynomial algebra. Then in order to prove Theorem 7.1, it suffices to consider the case where  $H^*(M; \mathbb{K})$  is a truncated polynomial algebra. Let  $\{\mathbb{E}_r^{*,*}, d_r\}$  be the EMSS converging to  $\mathbb{H}_{-*}(LM; \mathbb{K})$ . We first observe the following fact.

**Lemma 7.4.** *Suppose that  $H^*(M; \mathbb{K})$  is a truncated polynomial algebra generated by a single element. Then the EMSS  $\{\mathbb{E}_r^{*,*}, d_r\}$  collapses at the  $E_2$ -term.*

*Proof.* The proof of [28, Theorem 2.2] implies that the EMSS  $\{E_r, d_r\}$  collapses at the  $E_2$ -term and hence so does  $\{\mathbb{E}_r^{*,*}, d_r\}$ ; see also [28, Remark 2.6].  $\square$

We are left to compute the  $E_2$ -term and to solve all extension problems on  $\mathbb{E}_\infty^{*,*}$ .

Let  $\mathbb{K}$  be an arbitrary field and  $A$  a truncated polynomial algebra of the form  $\mathbb{K}[x]/(x^{n+1})$ , where  $|x| = 2m$ .

We recall here the calculations of the Hochschild cohomology ring of  $A$  due to Yang [47]. In what follows, let  $\text{ch}(\mathbb{K})$  stand for the characteristic of a field  $\mathbb{K}$ .

**Theorem 7.5** ([47, Theorems 4.6 , 4.7 and 4.8]). (i) *If  $n + 1 \not\equiv 0$  modulo  $\text{ch}(\mathbb{K})$ , then*

$$HH^*(A; A) \cong \mathbb{K}[x, u, t]/(x^{n+1}, u^2, x^n t, ux^n)$$

*as a graded algebra, where  $|x| = 2m$ ,  $|u| = 1$  and  $|t| = -2m(n + 1) + 2$ .*

(ii) *If  $\text{ch}(\mathbb{K}) \neq 2$  and  $n + 1 \equiv 0$  modulo  $\text{ch}(\mathbb{K})$ , then*

$$HH^*(A; A) \cong \mathbb{K}[x, v, t]/(x^{n+1}, v^2)$$

*as a graded algebra, where  $|x| = 2m$ ,  $|v| = -2m + 1$  and  $|t| = -2m(n + 1) + 2$ .*

(iii) *If  $\text{ch}(\mathbb{K}) = 2$  and  $n$  is odd, then*

$$HH^*(A; A) \cong \mathbb{K}[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1})$$

*as a graded algebra, where  $|x| = 2m$ ,  $|v| = -2m + 1$  and  $|t| = -2m(n + 1) + 2$ . Especially, when  $n = 1$ , as a graded algebra,*

$$HH^*(A; A) \cong \mathbb{K}[x, v, t]/(x^2, v^2 - t) \cong \wedge(x) \otimes \mathbb{K}[v].$$

*Remark 7.6.* In view of the 2-periodic resolution used in the proof of [47, Main Theorem], we see that  $\text{bideg } x = (0, 2m)$ ,  $\text{bideg } u = (1, 0)$ ,  $\text{bideg } v = (1, -2m)$  and  $\text{bideg } t = (2, -2m(n + 1))$  for the generators  $x$ ,  $u$ ,  $v$  and  $t$  in  $HH^*(A; A)$ ; see [47, Proposition 3.1] and the proofs of [47, Proposition 3.6] and [47, Theorem 4.7] for more details.

Let  $M$  be a simply-connected Poincaré duality space whose cohomology with coefficients in  $\mathbb{K}$  is isomorphic to  $A$  as an algebra.

**Theorem 7.7.** *If  $n + 1 \not\equiv 0$  modulo  $\text{ch}(\mathbb{K})$ , then*

$$\mathbb{H}_*(LM; \mathbb{K}) \cong \mathbb{K}[x, u, t]/(x^{n+1}, u^2, x^n t, ux^n)$$

*as a graded algebra, where  $|x| = -2m$ ,  $|u| = -1$  and  $|t| = 2m(n + 1) - 2$ .*

*Proof.* By virtue of Theorem 7.5(i), we have

$$\mathbb{E}_2^{*,*} \cong \mathbb{K}[x, u, t]/(x^{n+1}, u^2, x^n t, ux^n)$$

as a bigraded algebra, where  $\text{bideg } x = (0, 2m)$ ,  $\text{bideg } u = (1, 0)$  and  $\text{bideg } t = (2, -2m(n+1))$ ; see Remark 7.6 and Figure (7.1) below. Lemma 7.4 implies that, as bigraded algebras

$$\mathbb{E}_2^{p,q} \cong \mathbb{E}_\infty^{p,q} \cong Gr^{p,q}\mathbb{H}_*(LM) \cong F^p\mathbb{H}_{-(p+q)}(LM)/F^{p+1}\mathbb{H}_{-(p+q)}(LM).$$

In order to solve extension problems, we verify that the following equalities hold in  $\mathbb{H}_{-*}(LM; \mathbb{K})$ :

- (1)  $x^{n+1} = 0$ ,
- (2)  $u^2 = 0$ ,
- (3)  $x^n u = 0$ ,
- (4)  $x^n t = 0$ .

Since there exists no non-zero element in  $\mathbb{E}_2^{p,q}$  for  $p \geq 1$  and  $p+q = 2m(n+1)$ , it is readily seen that the equality (1) holds. We next verify that (2) holds. Suppose that  $u^2 = \sum \alpha_{ijk} x^i u^j t^k \neq 0$  for  $\alpha_{ijk} \in \mathbb{K}$ ,  $i < n+1$  and  $j = 0, 1$ . Since the total degrees of  $u^2$ ,  $x^i$  and  $t^k$  are even, it follows that  $j = 0$  and hence  $u^2 = \sum \alpha_{i0k} x^i t^k$ . We have

- $2 = 2mi + (-2m(n+1) + 2)k$ ,
- $2k \geq 3$

On the other hand, these deduce that

$$\begin{aligned} 0 &= 2mi - 2mk(n+1) + 2k - 2 \\ &< 2m(n+1) - 2mk(n+1) + 2k - 2 = 2(m(n+1) - 1)(1 - k) < 0, \end{aligned}$$

which is a contradiction. Thus the equality (2) holds. We see that (3) holds as well. In fact, suppose that  $x^n u = \sum \alpha_{ijk} x^i u^j t^k \neq 0$  for  $\alpha \in \mathbb{K}$  and  $i < n+1$ . For the same reason as above, we have  $j = 1$ ; that is,  $x^n u = \sum \alpha_{i1k} x^i u t^k$ . This implies that

- $2mn + 1 = 2mi + 1 + (-2m(n+1) + 2)k$ ,
- $1 + 2k \geq 2$ .

However these deduce that

$$\begin{aligned} 0 &= 2mi + 1 + (-2m(n+1) + 2)k - 2mn \\ &< 2m(n+1) + 1 + (-2m(n+1) + 2)k - 2mn \\ &= 2m(1 - k) + 2k(1 - mn) \leq 0. \end{aligned}$$

We thus obtain the equality (3). In order to verify that the equality (4) holds, we assume that  $x^n t = \sum \alpha_{ijk} x^i u^j t^k \neq 0$  for  $\alpha_{ijk} \in \mathbb{K}$  and  $i < n+1$ . It is readily seen that  $j = 0$  for dimensional reasons. This enables us to deduce that

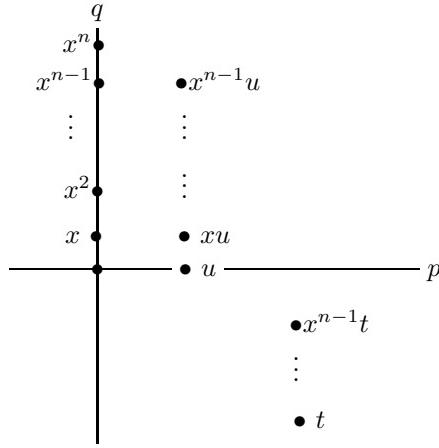
- $2mn - 2m(n+1) + 2 = 2mi + (-2m(n+1) + 2)k$ ,
- $2k \geq 3$ .

Since the natural number  $k$  is greater than or equal to 2, it follows that

$$\begin{aligned} 0 &= 2mi - 2mk(n+1) + 2k - 2mn + 2m(n+1) - 2 \\ &< 2m(n+1) - 2mk(n+1) + 2k - 2mn + 2m(n+1) - 2 \\ &= 2(1 - k)(mn - 1) + 2(2 - k)m \leq 0, \end{aligned}$$

which is a contradiction. Thus the equality (4) holds. This completes the proof.  $\square$

(7.1)



**Theorem 7.8.** *If  $n + 1 \equiv 0$  modulo  $\text{ch}(\mathbb{K})$ ,  $n + 1 \geq 3$  and  $\text{ch}(\mathbb{K}) \neq 2$ , then*

$$\mathbb{H}_*(LM; \mathbb{K}) \cong \mathbb{K}[x, v, t]/(x^{n+1}, v^2)$$

as a graded algebra, where  $|x| = -2m$ ,  $|v| = 2m - 1$  and  $|t| = 2m(n + 1) - 2$ .

*Proof.* In view of Theorem 7.5(ii), we have  $\mathbb{E}_2^{*,*} \cong \mathbb{K}[x, v, t]/(x^{n+1}, v^2)$  as a bigraded algebra, where  $\text{bideg } x = (0, 2m)$ ,  $\text{bideg } v = (1, -2m)$  and  $\text{bideg } t = (2, -2m(n + 1))$ ; see Figure (7.2) below. Lemma 7.4 yields that, as bigraded algebras

$$\mathbb{E}_2^{p,q} \cong \mathbb{E}_{\infty}^{p,q} \cong Gr^{p,q}\mathbb{H}_*(LM) \cong F^p\mathbb{H}_{-(p+q)}(LM)/F^{p+1}\mathbb{H}_{-(p+q)}(LM).$$

We verify that the following equalities hold in  $\mathbb{H}_{-*}(LM; \mathbb{K})$ :

- (1)  $x^{n+1} = 0$ ,
- (2)  $v^2 = 0$ .

By the same argument as in the proof of Theorem 7.9, it is readily seen that the equality (1) holds. Suppose that  $v^2 = \sum \alpha_{ijk} x^i v^j t^k \neq 0$  for  $\alpha_{ijk} \in \mathbb{K}$ ,  $i < n + 1$  and  $j = 0, 1$ . Since the total degrees of  $v^2$ ,  $x^i$  and  $t^k$  are even, we see that  $j = 0$  and hence  $v^2 = \sum \alpha_{i0k} x^i t^k$ . Thus an argument on the total degree and the filtration degree deduces that

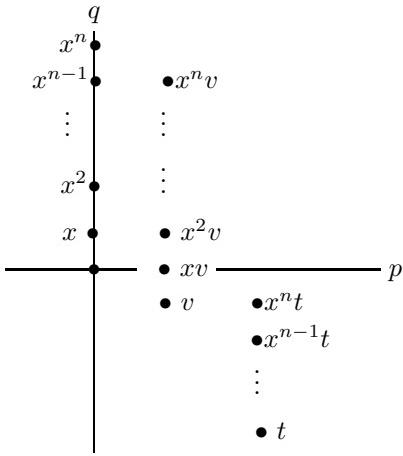
- $2 - 4m = 2mi + (-2m(n + 1) + 2)k$ ,
- $2k \geq 3$ .

Then we conclude that

$$\begin{aligned} 0 &= 2mi + (-2m(n + 1) + 2)k - 2 + 4m \\ &< 2m(n + 1) + (-2m(n + 1) + 2)k - 2 + 4m \\ &= -2(m(n + 1) - 1)(k - 1) + 4m \\ &\leq -2(3m - 1)(k - 1) + 4m \\ &\leq -2(3m - 1) + 4m = -2m + 2 \leq 0, \end{aligned}$$

which is a contradiction. This completes the proof.  $\square$

(7.2)



We next consider the case where  $\text{ch}(\mathbb{K}) = 2$ .

**Theorem 7.9.** *If  $n$  is odd,  $\text{ch}(\mathbb{K}) = 2$  and  $n + 1 \geq 3$ , then*

$$\mathbb{H}_*(LM; \mathbb{K}) \cong \mathbb{K}[x, v, t]/(x^{n+1}, v^2 - \frac{n+1}{2}tx^{n-1})$$

as a graded algebra, where  $|x| = -2m$ ,  $|v| = 2m - 1$  and  $|t| = 2m(n + 1) - 2$ .

*Proof.* The same argument as in the proof of Theorem 7.8 allows us to obtain the result.  $\square$

By considering the case where  $n = 1$  and  $\text{ch}(\mathbb{K}) = 2$ , namely the cohomology is an exterior algebra, we have Theorem 7.1.

Suppose that  $H^*(M; \mathbb{K})$  is an exterior algebra generated by a single element. For dimensional reasons, we see that the EMSS converging  $\{\mathbb{E}_r^{*,*}, d_r\}$  to the loop homology  $\mathbb{H}_{-*}(LM; \mathbb{K})$  collapses at the  $E_2$ -term and that there is no extension problem on the  $E_\infty$ -term. We then establish the following result.

**Theorem 7.10.** *Let  $M$  be a simply-connected space and  $\mathbb{K}$  an arbitrary field. Assume that  $H^*(M; \mathbb{K}) \cong \wedge(x)$ , where  $|x| = m$ . Then*

$$\mathbb{H}_*(LM; \mathbb{K}) \cong \wedge(x) \otimes \mathbb{K}[v]$$

as a graded algebra, where  $|x| = -m$  and  $|v| = m - 1$ .

*Proof of Theorem 7.1.* By virtue of Theorems 7.7, 7.8, 7.9 and 7.10, we have the result.  $\square$

*Remark 7.11.* We are aware that Theorems 7.7, 7.8, 7.9 and 7.10 recover the computations of the loop homology of spheres and complex projective spaces due to Cohen, Jones and Yan [8] when the coefficients of the homology are in a field.

## 8. A METHOD FOR SOLVING EXTENSION PROBLEMS ON THE EMSS FOR THE LOOP HOMOLOGY

In this section, we give a method for solving extension problems which appear in the first line  $\mathbb{E}_\infty^{0,*}$  of the EMSS converging to the loop homology of a Poincaré duality space.

Let  $ev_0 : LM \rightarrow M$  be the evaluation fibration over a simply-connected  $\mathbb{K}$ -Gorenstein space  $M$  of dimension  $d$ . Let  $s : M \rightarrow LM$  be the section of  $ev_0$  defined by  $s(x) = c_x$ , where  $c_x$  denotes the constant loop at  $x$ . We recall that the shifted homology  $\mathbb{H}_*(M) := H_{*+d}(M)$  is a commutative algebra with respect to the intersection pairing  $m$  defined by

$$m(a \otimes b) = (-1)^{d(|a|+d)}(\Delta^!)^\vee(a \otimes b).$$

**Proposition 8.1.** *The induced map  $s_* : \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$  is a morphism of algebras.*

We prove Proposition 8.1 after describing our main theorem (Theorem 8.2) in this section.

Let  $\{\mathbb{E}_r^{*,*}, d_r\}$  be the EMSS converging to the loop homology  $\mathbb{H}_*(LM)$ , which is described in Theorem 5.1. The following theorem is reliable when solving extension problems on the first line  $\mathbb{E}_\infty^{0,*}$ .

**Theorem 8.2.** *Let  $M$  be a simply-connected Poincaré duality space of dimension  $d$ . Then (i) there exists a first quadrant spectral sequence  $\{\tilde{\mathbb{E}}_r^{*,*}, \tilde{d}_r\}$  converging to the shifted homology  $\mathbb{H}_{-*}(M)$  as an algebra such that  $\tilde{\mathbb{E}}_r^{0,*} \cong H^*(M)$  as an algebra and  $\tilde{\mathbb{E}}_r^{i,*} = 0$  for  $i > 0$ .*

(ii) *There exists a morphism of spectral sequences*

$$\{s_{r*}\} : \{\tilde{\mathbb{E}}_r^{*,*}, \tilde{d}_r\} \rightarrow \{\mathbb{E}_r^{*,*}, d_r\}$$

such that (a) each  $s_{r*}$  is a morphism of bigraded algebras, (b) the diagram

$$\begin{array}{ccc} \tilde{\mathbb{E}}_2^{0,*} & \xrightarrow{s_{2*}} & \mathbb{E}_2^{0,*} \\ \cong \uparrow & & \uparrow \cong \\ H^*(M) & & \\ \parallel & & \\ H^*(\text{Hom}_{H^*(M)}(H^*(M), H^*(M))) & \xrightarrow[H(\text{Hom}(\varepsilon, 1))]{\quad} & HH^{0,*}(H^*(M), H^*(M)) \end{array}$$

is commutative, where  $H^*(M) \cong \tilde{\mathbb{E}}_2^{0,*}$  and  $\mathbb{E}_2^{*,*} \cong HH^*(H^*(M), H^*(M))$  are the isomorphisms in (i) and in Theorem 5.1, respectively and

(c) the map  $s_{\infty*}$  coincides with the composite

$$\tilde{\mathbb{E}}_\infty^{0,*} \cong \mathbb{H}_{-*}(M) \xrightarrow{s_*} F^0 \mathbb{H}_{-*}(LM) / F^1 \mathbb{H}_{-*}(LM) \cong \mathbb{E}_\infty^{0,*}.$$

**Remark 8.3.** The injective map  $ev_0^* : H^*(M) \rightarrow H^*(LM)$  factors through the edge homomorphism of the EMSS  $\{E_r^{*,*}, d_r\}$  converging to the cohomology  $H^*(LM)$ . Observe that the evaluation fibration  $p = ev_0 : LM \rightarrow M$  has a section. Thus we see that all the elements in the line  $E_2^{0,*}$  survive to the  $E_\infty$ -term. This implies that the elements in  $\mathbb{E}_2^{0,*}$  are permanent cycles.

**Remark 8.4.** The relative versions of Proposition 8.1 and Theorem 8.2 remain valid; that is, the spaces  $M$  and  $LM$  can be replaced with  $N$  and  $L_N M$ , respectively in the statements. Observe that in the relative version,  $\mathbb{E}_2^{*,*} \cong HH^*(H^*(M), H^*(N))$  as an algebra. This follows from the proofs mentioned below.

Before proving Proposition 8.1 and Theorem 8.2, we consider the following diagrams

$$(8.2) \quad \begin{array}{c} M = LM \times_M LM = M^I \times_M M^I \\ Comp \qquad \qquad \qquad Comp \\ M \xrightarrow{s} LM \xrightarrow{k} M^I \xleftarrow{u} M^3 \xrightarrow{v} M \\ M \xrightarrow{t} LM \xrightarrow{j} M^I \xleftarrow{v} M \\ M \xrightarrow{t} M^I \xleftarrow{u} M^3 \xrightarrow{v} M \\ M \xrightarrow{\Delta} M \times M \xleftarrow{\Delta} M, \end{array}$$

where  $t(x) = (c_x, c_x)$ . Observe that all squares in the diagrams are commutative.

*Proof of Proposition 8.1.* The commutativity of the left hand-side cube in (8.1) and Theorem 11.5 below enable us to deduce that  $H(\Delta^!) \circ t^* = (s \times s)^* \circ H(q^!)$ . By the commutativity of the left-hand side cube in (8.2), we see that  $t^* \circ \text{Comp}^* = s^*$  and hence  $H(\Delta^!) \circ s^* = (s \times s)^* H(q^!) \circ \text{Comp}^*$ . This implies that the induced map  $s_* : \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$  is a morphism of algebras.  $\square$

*Proof of Theorem 8.2.* The commutative diagrams (8.1) and (8.2) induce a commutative diagram

$$\begin{array}{ccccc}
& & \text{Tor}_{\Delta^*}(1, s^*) & & \\
& \nearrow EM & \downarrow \text{Tor}_{P_{13}^*}(1, c^*) & & \\
& \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^I)) & & & \\
& \searrow EM & & & \\
& \text{Tor}_{C^*(M^3)}^*(C^*(M), C^*(M^I \times_M M^I)) & & & \\
& \nearrow \text{Tor}_{v^*}(1, t^*) & & & \\
& \text{Tor}_{C^*(M)}^*(C^*(M), C^*(M)) & & & \\
& \nearrow \text{Tor}_{\Delta^*}(1, \Delta^*) & & & \\
& \text{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^I \times M^I)) & & & \\
& \nearrow \text{Tor}_{C^*(\Delta^2)}(1, C^*(s^2)) & & & \\
& \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^2)) & & & \\
& \nearrow \text{Tor}_1(\Delta^!, 1) & & & \\
& \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^I \times M^I)) & & & \\
& \nearrow \text{Tor}_{C^*(\Delta^2)}(1, C^*(s^2)) & & & \\
& \text{Tor}_{C^*(M^2)}^*(C^*(M^2), C^*(M^2)) & & & \\
& \nearrow \text{Tor}_1(\Delta^!, 1) & & & \\
H^*(M) & \xleftarrow{\cong} & H^*(LM) \times_M LM & \xleftarrow{\cong} & H^*(M) \\
& \nearrow s^* & \downarrow \text{Comp}^* & \nearrow t^* & \downarrow H(q^!) \\
& H^*(M) & & H^*(LM) & \\
& \nearrow \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^2)) & & \nearrow \text{Tor}_{C^*(M^3)}^*(C^*(M), C^*(M^I \times_M M^I)) & \\
& \nearrow \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^I \times M^I)) & & \nearrow \text{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2)) & \\
& \nearrow \text{Tor}_{C^*(\Delta^2)}(1, C^*(s^2)) & & \nearrow \text{Tor}_{C^*(\Delta^2)}(1, C^*(s^2)) & \\
& \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^2)) & & \text{Tor}_{C^*(M^2)}^*(C^*(M), C^*(M^2)) & \\
& \nearrow \text{Tor}_1(\Delta^!, 1) & & \nearrow \text{Tor}_1(\Delta^!, 1) & \\
H^*(\Delta^!) & \xleftarrow{\cong} & H^*(LM)^{\otimes 2} & \xleftarrow{\cong} & H^*(M)^{\otimes 2} \\
& \nearrow (s^*)^{\otimes 2} & & \nearrow \text{Tor}_{C^*(M^2)}^*(C^*(M^2), C^*(M^2)) & \\
H^*(M)^{\otimes 2} & \xleftarrow{\cong} & \text{Tor}_{C^*(M^2)}^*(C^*(M^2), C^*(M^2)). & &
\end{array}$$

The composite of the right hand-side vertical arrows in the big back square is the torsion functor description of  $Dlp$  in the proof of Theorem 2.3; see the diagram (3.3).

The Eilenberg-Moore map  $\text{Tor}_{C^*(M)}(C^*(M), C^*(M)) \xrightarrow{\cong} H^*(M)$  enables us to construct the EMSS converging to  $H^*(M)$ . Dualizing the EMSS, we have a spectral sequence  $\{\tilde{\mathbb{E}}_r^{*,*}, \tilde{d}_r\}$  converging to  $\mathbb{H}_*(M)$ . It is immediate that  $\tilde{\mathbb{E}}_r^{i,*} = 0$  for  $i > 0$ . The front cube (8.2) induces the top square in (8.3), which is commutative, and hence we obtain a morphism of spectral sequences  $\{s_{r*}\} : \{\tilde{\mathbb{E}}_r^{*,*}, \tilde{d}_r\} \rightarrow \{\mathbb{E}_r^{*,*}, d_r\}$ . Moreover by the commutativity of the diagram (8.3), we see that  $\{s_{r*}\}$  satisfies the conditions (ii)(a) and (ii)(c). In fact, the dual to the composite  $\sigma := \text{Tor}_1(\Delta^!, 1) \circ \text{Tor}_{\Delta^*}(1, \Delta^*)^{-1}$  gives rise to the product on each stage  $\tilde{\mathbb{E}}_r^{*,*}$ .

Let  $A$  denote the cohomology  $H^*(M)$ . In order to prove that  $s_{2*}$  satisfies (ii)(b), we consider a commutative diagram

$$(8.4) \quad \begin{array}{ccc} (\text{Tor}_{A \otimes 2}(A, A)^\vee)^{*+d} & \xleftarrow{\text{Tor}_m(1, 1)^\vee} & (\text{Tor}_A(A, A)^\vee)^{*+d} \\ \parallel & & \parallel \\ \text{Hom}_{\mathbb{K}}(H^{d-*}(A \otimes_{A \otimes 2} \mathbb{B}), \mathbb{K}) & \xleftarrow{\text{Hom}(H(1 \otimes \varepsilon), 1)} & \text{Hom}_{\mathbb{K}}(H^{d-*}(A \otimes_A A), \mathbb{K}) \\ \cong \downarrow & & \uparrow \cong \\ H^{*-d}(\text{Hom}_{\mathbb{K}}(A \otimes_{A \otimes 2} \mathbb{B}, \mathbb{K})) & \xleftarrow{H(\text{Hom}(1 \otimes \varepsilon, 1))} & H^{*-d}(\text{Hom}_{\mathbb{K}}(A \otimes_A A, \mathbb{K})) \\ \iota_* \downarrow & & \uparrow \iota_* \\ H^{*-d}(\text{Hom}_{A \otimes 2}(\mathbb{B}, A^\vee)) & \xleftarrow{H(\text{Hom}(\varepsilon, 1))} & H^{*-d}(\text{Hom}_A(A, A^\vee)) \\ \theta_{A*} \downarrow & & \uparrow \theta_{A*} \\ H^*(\text{Hom}_{A \otimes 2}(\mathbb{B}, A)) & \xleftarrow{H(\text{Hom}(\varepsilon, 1))} & H^*(\text{Hom}_A(A, A)) \\ \parallel & & \parallel \\ HH^*(A, A) & \xleftarrow{H(\text{Hom}(\varepsilon, 1))} & A. \end{array}$$

Observe that the composite of the left hand-side vertical arrows is nothing but the isomorphism  $\zeta$  in Theorem 5.3. Thus we obtain the commutative diagram in (ii)(b). We are left to prove that  $\tilde{\mathbb{E}}_2^{0,*} \cong H^*(M)$  as an algebra. Let  $\tilde{\zeta}$  be the composite of the right hand-side vertical arrows in (8.4). Consider the following diagram

$$(8.5) \quad \begin{array}{ccccc} HH^*(A, A) \otimes HH^*(A, A) & \xrightarrow{\cup} & HH^*(A, A) & & \\ \zeta \otimes \zeta \downarrow \cong & & \cong \downarrow \zeta & & \\ \text{Tor}_{A \otimes 2}(A, A)^\vee \otimes \text{Tor}_{A \otimes 2}(A, A)^\vee & \xrightarrow{(Dlp_A)^\vee} & \text{Tor}_{A \otimes 2}(A, A)^\vee & & \\ \xi \otimes \xi \uparrow & & \xi \uparrow & & \\ \text{Tor}_A(A, A)^\vee \otimes \text{Tor}_A(A, A)^\vee & \xrightarrow{\mu} & \text{Tor}_A(A, A)^\vee & & \\ \tilde{\zeta} \otimes \tilde{\zeta} \cong \uparrow & & \cong \uparrow \tilde{\zeta} & & \\ A \otimes A & \xrightarrow{\cup} & A & & \end{array}$$

in which squares are commutative except for the lower one, where  $\eta = \text{Hom}(\varepsilon, 1) : A = H^*(\text{Hom}_A(A, A)) \rightarrow HH^*(A, A)$ ,  $\xi = \text{Tor}_m(1, 1)^\vee$  and  $\mu$  denotes the dual to the map induced by the composite  $\sigma$  mentioned above. The map  $\text{Tor}_m(1, 1)$  is an epimorphism and hence  $\xi$  is a monomorphism. By using the commutativity of the diagram (8.5), we see that the lower square is also commutative. This completes the proof.  $\square$

With the aid of the spectral sequence in Theorem 2.11, we show that the relative loop homology of a Poincaré duality space is unital.

**Proposition 8.5.** *Let  $N$  be a simply-connected Poincaré duality space of dimension  $d$ . Then the loop homology  $\mathbb{H}_*(L_N M)$  is unital.*

*Proof.* In the rational case, the result follows from Theorem 2.17; see also [16, Theorem 1]. We assume that the characteristic of the underlying field is positive.

Let  $1_N$  stand for the unit of the intersection homology  $\mathbb{H}_*(N)$ , namely the fundamental class of  $N$ . Let  $s_* : \mathbb{H}_*(N) \rightarrow \mathbb{H}_*(L_N M)$  be the algebra map mentioned in Proposition 8.1. We put  $\mathbb{I} = s_*(1_N)$ . Then it is immediate that  $\mathbb{I} \cdot \mathbb{I} = \mathbb{I}$ .

Recall the right half-plane spectral sequence  $\{\mathbb{E}_r^{*,*}, d_r\}$  described in Theorem 2.11. It follows from Remark 8.3 that the unit  $1$  in the bigraded algebra  $\mathbb{E}_2^{*,*} \cong HH^{*,*}(H^*(M), H^*(N))$  is a permanent cycle. Observe that the Hochschild cohomology is unital; see (5.1). In view of Theorem 8.2 (ii)(b) and (c), we can choose  $\mathbb{I}$  as a representative of the unit. In fact the diagram (8.4) enables us to deduce that  $s_{2*}$  sends the fundamental class to the unit  $1$  in  $\mathbb{E}_2^{*,*}$  up to isomorphism.

Let  $\{F^p\}_{p \geq 0}$  be the filtration of the loop homolog  $\mathbb{H}_*(L_N M)$  which the spectral sequence  $\{\mathbb{E}_r^{*,*}, d_r\}$  provides. Then we see that  $(F^p)^n = 0$  for  $p > \dim N - n$ ; see Remark 5.5. This yields that  $\mathbb{I} \cdot a = a$  for any  $a$  in  $(F^p)^n$  with  $p = \dim N - n$ . Suppose that  $\mathbb{I} \cdot Q = Q$  for any  $Q \in (F^{>s})^n$ . Let  $\alpha$  be an element in  $(F^s)^n$ . Since  $\mathbb{I} \cdot \alpha = \alpha$  in  $\mathbb{E}_\infty^{s,*}$ , it follows that  $\mathbb{I} \cdot \alpha = \alpha + R$  for some  $R$  in  $(F^{s+1})^n$  and hence

$$\mathbb{I} \cdot \alpha = (\mathbb{I} \cdot \mathbb{I}) \cdot \alpha = \mathbb{I} \cdot (\mathbb{I} \cdot \alpha) = \mathbb{I} \cdot \alpha + R = \alpha + 2R.$$

Iterating the multiplication by  $\mathbb{I}$ , we see that  $\mathbb{I} \cdot \alpha = \alpha + ch(\mathbb{K})R = \alpha$ . This completes the proof.  $\square$

We now give an application of Theorem 8.2.

**Theorem 8.6.** *Let  $M$  be the Stiefel manifold  $SO(m+n)/SO(n)$ . Suppose that  $m \leq \min\{4, n\}$ . Then*

$$\mathbb{H}_*(LM; \mathbb{Z}/2) \cong \wedge(x_n, x_{n+1}, \dots, x_{n+m-1}) \otimes \mathbb{Z}/2[\nu_n^*, \nu_{n+1}^*, \dots, \nu_{n+m-1}^*]$$

as an algebra, where  $\deg x_i = -i$  and  $\deg \nu_j^* = -(1-j)$ .

We mention that Chataur and Le Borgne [5] have determined the loop homology of  $SO(2+n)/SO(n)$  with coefficients in  $\mathbb{Z}$  by using enriched Leray-Serre and Morse spectral sequences with the loop product; see [5, Section 2] and [31, Theorem 2].

*Proof of Theorem 8.6.* Consider the EMSS  $\{\mathbb{E}_r^{*,*}, d_r\}$  converging to  $\mathbb{H}_*(LM)$ . Since  $m \leq n$ , it follows that  $H^*(M; \mathbb{Z}/2) \cong \wedge(x_n, x_{n+1}, \dots, x_{n+m-1})$  as an algebra. Moreover, the condition that  $m \leq 4$  and the proof of [24, Corollary 5 (1)] imply that  $\{\mathbb{E}_r^{*,*}, d_r\}$  collapses at the  $E_2$ -term; see also [24, Proposition 1.7 (2)] and the proof of [24, Theorem 4]. By virtue of [25, Proposition 2.4], we see that as a bigraded algebra,

$$\mathbb{E}_\infty^{*,*} \cong \wedge(x_n, x_{n+1}, \dots, x_{n+m-1}) \otimes \mathbb{Z}/2[\nu_n^*, \nu_{n+1}^*, \dots, \nu_{n+m-1}^*],$$

where  $\text{bideg } x_i = (0, i)$  and  $\text{bideg } \nu_i^* = (1, -i)$ .

We solve the extension problems in the  $E_\infty$ -term. Recall the spectral sequences and the morphism  $\{s_{r*}\}$  of spectral sequences in Theorem 8.2. It follows from Theorem 8.2(ii)(b) and (c) that for the induced map  $s_* : \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$ ,  $s_*(x_i) = x_i$  for any  $1 \leq i \leq n+m-1$ . Observe that  $s_*$  is a morphism of algebras;

see Proposition 8.1. It turns out that  $x_i^2 = 0$  in  $\mathbb{H}_*(LM)$  for any  $i$ . We have the result.  $\square$

*Remark 8.7.* Let  $X$  be a simply-connected space whose mod  $p$  cohomology is an exterior algebra, say  $H^*(X; \mathbb{Z}/p) \cong \wedge(y_1, \dots, y_l)$ . Suppose that either of the following conditions (I) and (II) holds.

- (I)  $X$  is an H-space and  $\deg y_i$  is odd for any  $i$ .
- (II)  $Sq^1 \equiv 0$  if  $p = 2$ .

Then the same argument as in the proof of Theorem 8.6 enables us to conclude that

$$\mathbb{H}_*(LX) \cong \wedge(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_l) \otimes \mathbb{Z}/p[\nu_1^*, \nu_2^*, \dots, \nu_l^*]$$

as an algebra, where  $\deg \tilde{y}_j = -\deg y_j$  and  $\deg \nu_j^* = \deg y_j - 1$ .

A more general result will appear in [27].

## 9. NATURALITY AND COMPUTATIONS IN THE RELATIVE CASE

Let  $\mathbf{Poincaré}_M$  be the category of simply-connected based Poincaré duality spaces over  $M$  and based maps; that is, a morphism from  $\alpha_1 : N_1 \rightarrow M$  to  $\alpha_2 : N_2 \rightarrow M$  is a based map  $f : N_1 \rightarrow N_2$  with  $\alpha_1 = \alpha_2 \circ f$ . Let  $\mathbf{Top}_1^N$  be the category of simply-connected spaces under  $N$ . We denote by  $\mathbf{GradedAlg}_A$  and  $\mathbf{GradedAlg}^A$  the categories of unital graded algebras over an algebra  $A$  and of those under  $A$ , respectively. Assume that  $N$  is a simply-connected Poincaré duality spaces. Then the loop homology  $\mathbb{H}_*(L_f M) := H_{*+\dim s(f)}(L_f M)$  comes with the loop product, where  $s(f) = N$ ; see Remark 2.6. In consequence, our consideration in this paper permits us to deduce the following remarkable theorem.

**Theorem 9.1.** (1) *The loop homology gives rise to functors*

$$\mathbb{H}_*(L?M) := H_{*+\dim s(?)}(L?M) : \mathbf{Poincaré}_M^{op} \rightarrow \mathbf{GradedAlg}_{H_*(\Omega M)}$$

and

$$\mathbb{H}_*(L_N?) := H_{*+\dim N}(L_N?) : \mathbf{Top}_1^N \rightarrow \mathbf{GradedAlg}^{\mathbb{H}_*(N)}.$$

Suppose further that  $M$  is a simply-connected Poincaré duality space. Then one has a functor

$$\mathbb{H}_*(L?M) : \mathbf{Poincaré}_M^{op} \rightarrow \mathbf{GradedAlg}_{H_*(\Omega M)}^{\mathbb{H}_*(LM)}.$$

Here  $\mathbf{GradedAlg}_{H_*(\Omega M)}^{\mathbb{H}_*(LM)}$  denotes the category of unital graded algebras over the algebra  $H_*(\Omega M)$  with the Pontrjagin product and under the loop homology  $\mathbb{H}_*(LM)$ .

(2) *The multiplicative spectral sequence in Theorem 2.11 converging to the relative loop homology is natural with respect to morphisms in  $\mathbf{Poincaré}_M$  and  $\mathbf{Top}_1^N$ ; that is, for any morphism  $\rho$  in  $\mathbf{Poincaré}_M$  or  $\mathbf{Top}_1^N$ , there exists a multiplicative morphism of the spectral sequences such that the map between the associated bigraded algebras, which  $\mathbb{H}_*(L_N?)(\rho)$  or  $\mathbb{H}_*(L?M)(\rho)$  gives rise to, coincides with the map on the  $E_\infty$ -terms up to isomorphism.*

If  $N$  is a closed oriented smooth manifold, part 1) follows easily from [19, Theorem 8, see also Corollary 9 and Proposition 10]. Using [15, Theorem 4], it is easy to extend Theorem 8 of [19] to Poincaré duality space. Therefore 1) can be proved easily. But in order to prove part 2), we need to interpret 1) in term of differential torsion product (See the proof of Propositions 9.2 and 9.4).

For a map  $f : N \rightarrow M$  between simply-connected Poincaré duality spaces, Theorem 9.1 enables one to obtain algebra maps

$$\mathbb{H}_*(L_N N) \xrightarrow{\mathbb{H}_*(L_N ?)(f)} \mathbb{H}_*(L_N M) \xleftarrow{\mathbb{H}_*(L ? M)(f)} \mathbb{H}_*(L_M M).$$

These maps provide tools to overcome the difficulty arising from the lack of functoriality in the loop homology. For example, if  $f$  is a smooth orientation preserving homotopy equivalence between manifolds, in [19, Proposition 23], Gruher and Salvatore showed that these two algebras maps are isomorphisms and that their composite coincides with  $\mathbb{H}_*(Lf) : \mathbb{H}_*(LN) \rightarrow \mathbb{H}_*(LM)$ . Here we give a computational example by using functors  $\mathbb{H}_*(L_N ?)$  and  $\mathbb{H}_*(L ? M)$ .

In order to prove Theorem 9.1, we give a correspondence of morphisms between the categories  $\mathbf{Poincaré}_M^{op}$  and  $\mathbf{GradedAlg}_{H_*(\Omega M)}$ . We will describe the proof in terms of the derived tensor products  $- \otimes^{\mathbb{L}} -$ .

Let  $M$  be a space and  $N_1, N_2$  Poincaré duality spaces of dimension  $d_1$  and  $d_2$ , respectively. For a morphism

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ \alpha_1 \searrow & & \swarrow \alpha_2 \\ & M & \end{array}$$

in  $\mathbf{Poincaré}_M$ , we have a commutative diagram

$$\begin{array}{ccccccc} & & L_{\alpha_2} M & \longrightarrow & LM & \longrightarrow & M^I \\ & F \nearrow & ev_0 \downarrow & & ev_0 \downarrow & & (ev_0, ev_1) \downarrow \\ L_{\alpha_1} M & \longrightarrow & N_2 & \xrightarrow{\alpha_2} & M & \xrightarrow{\Delta} & M \times M \\ ev_0 \downarrow & f \nearrow & & \nearrow \alpha_1 & & & \\ N_1 & & & & & & \end{array}$$

for which back squares are pull-back diagrams. The singular cochain algebra  $C^*(N_i)$  is considered  $C^*(M^2)$ -module structure via the map  $\alpha_i^* \Delta^*$ . By Theorem 2.1, we obtain a right  $C^*(M^2)$ -module map

$$f^! : \mathbb{B}_1 \longrightarrow \mathbb{B}_2$$

with degree  $d_2 - d_1$ . Here  $\mathbb{B}_i$  is a right  $C^*(M^2)$ -semifree resolution of  $C^*(N_i)$ . Then, we define a map  $F^! : H^*(L_{\alpha_1} M) \rightarrow H^*(L_{\alpha_2} M)$  to be the composite

$$\begin{array}{c} H^*(L_{\alpha_1} M) \xrightarrow[\cong]{EM^{-1}} H^*(\mathbb{B}_1 \otimes_{C^*(M^2)} F) \\ \downarrow H(f^! \otimes 1) \\ H^*(\mathbb{B}_2 \otimes_{C^*(M^2)} F) \xrightarrow[\cong]{EM} H^*(L_{\alpha_2} M), \end{array}$$

where  $\varepsilon : F \rightarrow C^*(M^I)$  is a left  $C^*(M^2)$ -semifree resolution of  $C^*(M^I)$ .

**Proposition 9.2.** (i) *The shriek map of  $F : L_{\alpha_1}M \rightarrow L_{\alpha_2}M$  is compatible with the dual loop product, that is, the following diagram is commutative*

$$\begin{array}{ccc} H^*(L_{\alpha_1}M) & \xrightarrow{F^!} & H^*(L_{\alpha_2}M) \\ Dlp \downarrow & & \downarrow Dlp \\ H^*(L_{\alpha_1}M) \otimes H^*(L_{\alpha_1}M) & \xrightarrow[(-1)^{d_1(d_2-d_1)}]{} & H^*(L_{\alpha_2}M) \otimes H^*(L_{\alpha_2}M). \end{array}$$

(ii) *The square*

$$\begin{array}{ccc} (\mathrm{Tor}_{H^*(M^2)}(H^*(N_2), H^*(M)))^\vee & \xleftarrow{(-1)^{d_2}\Theta} & HH^*(H^*(M), H^*(N_2)) \\ (-1)^{d_1(d_2-d_1)}\mathrm{Tor}_1(H(f^!), 1)^\vee \downarrow & & \downarrow HH(1, f^*) \\ (\mathrm{Tor}_{H^*(M^2)}(H^*(N_1), H^*(M)))^\vee & \xleftarrow{(-1)^{d_2}\Theta} & HH^*(H^*(M), H^*(N_1)) \end{array}$$

is commutative.

*Proof.* (i) By a relative version of Theorem 2.3, we see that the composite

$$\begin{array}{ccccc} C^*(N_i) \otimes_{C^*(M^2)}^{\mathbb{L}} C^*(M^I) & \xrightarrow{p_{13}^* \otimes_{p_{13}^*} c^*} & C^*(N_i) \otimes_{C^*(M^3)}^{\mathbb{L}} C^*(M^I \times_M M^I) & & \\ & & \simeq \downarrow \omega^* \otimes_{\omega^*} \tilde{q}^* & & \\ & & C^*(N_i) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I) & & \\ & & \downarrow \Delta^! \otimes 1 & & \\ & & C^*(N_i^2) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I) & & \\ & & \downarrow \mathrm{EZ}^\vee \otimes_{\mathrm{EZ}^\vee} \mathrm{EZ}^\vee & & \\ & & (C_*(N_i)^{\otimes 2})^\vee \otimes_{(C_*(M^2)^{\otimes 2})^\vee} (C_*(M^I)^{\otimes 2})^\vee & & \\ & & \simeq \downarrow \Theta \otimes_{\Theta} \Theta & & \\ (C^*(N_i) \otimes_{C^*(M^2)}^{\mathbb{L}} C^*(M^I))^{\otimes 2} & \xrightarrow[\cong]{\top} & C^*(N_i)^{\otimes 2} \otimes_{C^*(M^2)^{\otimes 2}}^{\mathbb{L}} C^*(M^I)^{\otimes 2} & & \end{array}$$

induces the dual loop product of  $H^*(L_{\alpha_i}M)$ . Since the morphism  $f^!$  is in  $D(\mathrm{Mod}-C^*(M))$ , it follows that  $f^!$  is considered a morphism in  $D(\mathrm{Mod}-C^*(M^i))$  via  $p_{13}^*$  and  $\omega^*$  for  $i = 3, 4$ . This enables us to obtain a homotopy commutative diagram

$$(9.1) \quad \begin{array}{ccccc} C^*(N_1) \otimes_{C^*(M^2)}^{\mathbb{L}} C^*(M^I) & \xrightarrow{f^! \otimes 1} & C^*(N_2) \otimes_{C^*(M^2)}^{\mathbb{L}} C^*(M^I) & & \\ p_{13}^* \otimes_{p_{13}^*} c^* \downarrow & & \downarrow p_{13}^* \otimes_{p_{13}^*} c^* & & \\ C^*(N_1) \otimes_{C^*(M^3)}^{\mathbb{L}} C^*(M^I \times_M M^I) & \xrightarrow{f^! \otimes 1} & C^*(N_2) \otimes_{C^*(M^3)}^{\mathbb{L}} C^*(M^I \times_M M^I) & & \\ \omega^* \otimes_{\omega^*} \tilde{q}^* \uparrow \simeq & & \uparrow \omega^* \otimes_{\omega^*} \tilde{q}^* & & \\ C^*(N_1) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I) & \xrightarrow{f^! \otimes 1} & C^*(N_2) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I) & & \\ \Delta^! \otimes 1 \downarrow & & \downarrow \Delta^! \otimes 1 & & \\ C^*(N_1^2) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I) & \xrightarrow{(f \times f)^! \otimes 1} & C^*(N_2^2) \otimes_{C^*(M^4)}^{\mathbb{L}} C^*(M^I \times M^I). & & \end{array}$$

The fact that  $\Delta^! f^!$  is homotopic to  $(f \times f)^! \Delta^!$  deduces the commutativity of the bottom square of the diagram (9.1). By Theorem 11.6 (1) and (2), we see that there is a  $(C_*(M)^{\otimes 2})^\vee$ -module map  $h$  such that the diagrams

$$\begin{array}{ccccc}
C^*(N_1^2) \otimes_{C^*(M^4)}^\mathbb{L} C^*(M^I \times M^I) & \xrightarrow{(f \times f)^! \otimes 1} & C^*(N_2^2) \otimes_{C^*(M^4)}^\mathbb{L} C^*(M^I \times M^I) \\
\text{EZ}^\vee \otimes_{\text{EZ}^\vee} \text{EZ}^\vee \downarrow & & \downarrow \text{EZ}^\vee \otimes_{\text{EZ}^\vee} \text{EZ}^\vee \\
(C_*(N_1)^{\otimes 2})^\vee \otimes_{(C_*(M^2)^{\otimes 2})^\vee}^\mathbb{L} (C_*(M^I)^{\otimes 2})^\vee & \xrightarrow{h \otimes 1} & (C_*(N_2)^{\otimes 2})^\vee \otimes_{(C_*(M^2)^{\otimes 2})^\vee}^\mathbb{L} (C_*(M^I)^{\otimes 2})^\vee \\
\Theta \otimes_\Theta \Theta \uparrow & & \uparrow \Theta \otimes_\Theta \Theta \\
C^*(N_1)^{\otimes 2} \otimes_{C^*(M)^{\otimes 2}}^\mathbb{L} C^*(M^I)^{\otimes 2} & \xrightarrow{(-1)^{d_1(d_2-d_1)} (f^! \otimes f^!) \otimes (1 \otimes 1)} & C^*(N_2)^{\otimes 2} \otimes_{C^*(M)^{\otimes 2}}^\mathbb{L} C^*(M^I)^{\otimes 2} \\
\top \uparrow & & \uparrow \top \\
(C^*(N_1) \otimes_{C^*(M^2)}^\mathbb{L} C^*(M^I))^{\otimes 2} & \xrightarrow{(-1)^{d_1(d_2-d_1)} (f^! \otimes 1)^{\otimes 2}} & (C^*(N_2) \otimes_{C^*(M^2)}^\mathbb{L} C^*(M^I))^{\otimes 2}
\end{array}$$

are homotopy commutative. Therefore, we have (i) by combining the commutative squares mentioned above.

(ii) We recall the isomorphism  $\theta_i : \mathbb{H}_*(N_i) \rightarrow \mathbb{H}_*(N_i)^\vee$  which Poincaré duality on  $H^*(N_i)$  gives; see Definition 5.2 and the ensuing discussion. Since the diagram

$$\begin{array}{ccc}
H^*(N_1) & \xrightarrow{H(f^!)} & H^*(N_2) \\
\theta_1 \downarrow & & \downarrow \theta_2 \\
H^*(N_1)^\vee & \xrightarrow{H(f)^{\vee}} & H^*(N_2)^\vee,
\end{array}$$

is commutative, by Lemma 11.9, we have a commutative diagram

$$\begin{array}{ccccc}
H^*(N_1)^\vee & \xleftarrow{H(f^!)^\vee} & H^*(N_2)^\vee & & \\
\theta_1^\vee \uparrow & & \uparrow \theta_2^\vee & & \\
H^*(N_1)^{\vee\vee} & \xleftarrow{(-1)^{d_2(d_2-d_1)} H(f)^{\vee\vee}} & H^*(N_2)^{\vee\vee} & & \\
\cong \uparrow & & \uparrow \cong & & \\
H^*(N_1) & \xleftarrow{(-1)^{d_2(d_2-d_1)} H(f)} & H^*(N_2), & &
\end{array}$$

which is the dual to the above square. Therefore, the commutativity of the diagram

$$\begin{array}{ccc}
HH^*(H^*(M), H^*(N_2)) & \xrightarrow{(-1)^{d_2(d_2-d_1)} HH(1, H(f))} & HH^*(H^*(M), H^*(N_1)) \\
\text{Hom}_1(1, \theta_2) \downarrow & & \downarrow \text{Hom}_1(1, \theta_1) \\
H(\text{Hom}_{H^*(M^2)}(\mathbb{B}, H^*(N_2)^\vee)) & \xrightarrow{H(\text{Hom}_1(1, (H(f^!))^\vee))} & H(\text{Hom}_{H^*(M^2)}(\mathbb{B}, H^*(N_1)^\vee)) \\
\iota_* \downarrow & & \downarrow \iota_* \\
H(\text{Hom}_{\mathbb{K}}(H^*(N_2) \otimes_{H^*(M^2)} \mathbb{B}), \mathbb{K}) & \xrightarrow{H(\text{Hom}_1(H(f^!) \otimes 1, 1))} & H(\text{Hom}_{\mathbb{K}}(H^*(N_1) \otimes_{H^*(M^2)} \mathbb{B}), \mathbb{K}) \\
\cong \downarrow & & \downarrow \cong \\
(H(H^*(N_2) \otimes_{H^*(M^2)} \mathbb{B}))^\vee & \xrightarrow{(H(f^!) \otimes 1)^\vee} & (H(H^*(N_1) \otimes_{H^*(M^2)} \mathbb{B}))^\vee \\
\cong \downarrow & & \downarrow \cong \\
(\text{Tor}_{H^*(M^2)}(H^*(N_2), H^*(M)))^\vee & \xrightarrow{\text{Tor}_1(H(f^!), 1)} & (\text{Tor}_{H^*(M^2)}(H^*(N_1), H^*(M)))^\vee.
\end{array}$$

implies the assertion (ii):  $\square$

We recall the product  $m_i$  on the loop homology  $\mathbb{H}_*(L_{\alpha_i}M) = H_{*+d_i}(L_{\alpha_i}M)$  defined by

$$m_i(a \otimes b) = (-1)^{d_i(|a|+d_i)}(Dlp)^\vee \eta(a \otimes b),$$

where  $\eta : H_*(L_{\alpha_i}M)^{\otimes 2} \cong (H^*(L_{\alpha_i}M)^\vee)^{\otimes 2} \rightarrow (H^*(L_{\alpha_i}M)^{\otimes 2})^\vee$  is the natural isomorphism.

**Proposition 9.3.** *The map  $\widetilde{F}^! = (-1)^{d_1(d_2-d_1)}(F^!)^\vee : \mathbb{H}_*(L_{\alpha_2}M) \rightarrow \mathbb{H}_*(L_{\alpha_1}M)$  is an algebra map.*

*Proof.* For an element  $a \otimes b$  in  $\mathbb{H}_*(L_{\alpha_2}M)^{\otimes 2}$ , since  $Dlp \circ F^! = (-1)^{d_1(d_2-d_1)}(F^! \otimes F^!) \circ Dlp$ , it follows that  $(F^!)^\vee(Dlp)^\vee = (-1)^{d_1(d_2-d_1)+d_2(d_2-d_1)}(Dlp)^\vee(F^! \otimes F^!)^\vee$ ; see Lemma 11.9 for the sign. Then, we see that

$$\begin{aligned} & m_1(\widetilde{F}^! \otimes \widetilde{F}^!)(a \otimes b) \\ &= m_1((F^!)^\vee(a) \otimes (F^!)^\vee(b)) \\ &= (-1)^{d_1(|a|+d_2-d_1+d_1)}(Dlp)^\vee \eta((F^!)^\vee(a) \otimes (F^!)^\vee(b)) \\ &= (-1)^{d_1(|a|+d_2)+|a|(d_2-d_1)}(Dlp)^\vee(F^! \otimes F^!)^\vee \eta(a \otimes b) \\ &= (-1)^{d_1(|a|+d_2)+|a|(d_2-d_1)+d_1(d_2-d_1)+d_2(d_2-d_1)}(F^!)^\vee(Dlp)^\vee \eta(a \otimes b) \\ &= (-1)^{d_1 d_2 - d_1 + d_2 (|a|+d_2)}(F^!)^\vee(Dlp)^\vee \eta(a \otimes b) \\ &= \widetilde{F}^! m_2(a \otimes b). \end{aligned}$$

This completes the proof.  $\square$

Let  $N$  be a simply-connected Poincaré duality space of dimension  $d$ . Consider the commutative diagram of simply-connected spaces

$$\begin{array}{ccc} M_1 & \xrightarrow{g} & M_2 \\ & \swarrow \beta_1 & \nearrow \beta_2 \\ & N. & \end{array}$$

Let  $\tilde{g}$  denote  $H^*(L_N g)$ , the morphism induced in cohomology by  $L_N g : L_N M_1 \rightarrow L_N M_2$ . Then the map  $\tilde{g}$  coincides with the composite

$$H^*(L_{\beta_2} M_2) \xrightarrow{\text{EM}^{-1}} H^*(\mathbb{B}'_2 \otimes_{C^*(M_2^2)} F_2) \xrightarrow{\overline{id} \otimes_{(g^2)^*} (\overline{g^I})^*} H^*(\mathbb{B}'_1 \otimes_{C^*(M_1^2)} F_1) \xrightarrow{\text{EM}} H^*(L_{\beta_1} M_1).$$

Here  $\mathbb{B}'_i$  is a right  $C^*(M_i^2)$ -semifree resolution of  $C^*(N)$ ,  $F_i$  is a left  $C^*(M_i^2)$ -semifree resolution of  $C^*(M_i^I)$ . Moreover,  $\overline{id}$  and  $(\overline{g^I})^*$  are the maps which make the diagrams

$$\begin{array}{ccc} \mathbb{B}'_2 & \xrightarrow{\overline{id}} & \mathbb{B}'_1 \\ & \xrightarrow{\varepsilon} & \downarrow \simeq \varepsilon \\ & C^*(N), & \end{array} \quad \begin{array}{ccc} F_2 & \xrightarrow{\varepsilon} & F_1 \\ \xrightarrow{\simeq} & C^*(M_2^I) & \xrightarrow{(g^I)^*} \\ & C^*(M_1^I) & \downarrow \simeq \varepsilon \end{array}$$

commutative up to homotopy. Observe that  $\overline{id} = id$  and  $(\overline{g^I})^* = (g^I)^*$  in the derived categories  $D(\text{Mod-}C^*(M_2^2))$  and  $D(C^*(M_2^2)\text{-Mod})$ , respectively.

**Proposition 9.4.** (i) *The map  $\tilde{g}$  is compatible with the dual loop product.*  
(ii) *The dual  $(\tilde{g})^\vee = H_*(L_N g) : \mathbb{H}_*(L_{\beta_1} M_1) \rightarrow \mathbb{H}_*(L_{\beta_2} M_2)$  is an algebra map.*

*Proof.* Theorem 11.6 (1) and (2) enable us to obtain the commutative squares up to homotopy

$$\begin{array}{ccccc}
C^*(N) \otimes_{C^*(M_2^2)}^{\mathbb{L}} C^*(M_2^I) & \xrightarrow{1 \otimes_{(g^2)^*} (g^I)^*} & C^*(N) \otimes_{C^*(M_1^2)}^{\mathbb{L}} C^*(M_1^I) \\
p_{13}^* \otimes_{p_{13}^*} c^* \downarrow & & \downarrow p_{13}^* \otimes_{p_{13}^*} c^* \\
C^*(N) \otimes_{C^*(M_2^3)}^{\mathbb{L}} C^*(M_2^I \times_{M_2} M_2^I) & \xrightarrow{1 \otimes_{(g^3)^*} (g^I \times_g g^I)^*} & C^*(N) \otimes_{C^*(M_1^3)}^{\mathbb{L}} C^*(M_1^I \times_{M_1} M_1^I) \\
\omega^* \otimes_{\omega^*} \tilde{q}^* \uparrow & & \uparrow \omega^* \otimes_{\omega^*} \tilde{q}^* \\
C^*(N) \otimes_{C^*(M_2^4)}^{\mathbb{L}} C^*(M_2^I \times M_2^I) & \xrightarrow{1 \otimes_{(g^4)^*} (g^I \times_g g^I)^*} & C^*(N) \otimes_{C^*(M_1^4)}^{\mathbb{L}} C^*(M_1^I \times M_1^I) \\
\Delta' \otimes 1 \downarrow & & \downarrow \Delta' \otimes 1 \\
C^*(N^2) \otimes_{C^*(M_2^4)}^{\mathbb{L}} C^*(M_2^I \times M_2^I) & \xrightarrow{1 \otimes_{(g^4)^*} (g^I \times_g g^I)^*} & C^*(N^2) \otimes_{C^*(M_1^4)}^{\mathbb{L}} C^*(M_1^I \times M_1^I) \\
EZ^\vee \otimes_{EZ^\vee} EZ^\vee \downarrow & & \downarrow EZ^\vee \otimes_{EZ^\vee} EZ^\vee \\
(C_*(N)^{\otimes 2})^\vee \otimes_{(C_*(M_2^2)^{\otimes 2})^\vee}^{\mathbb{L}} (C_*(M_2^I)^{\otimes 2})^\vee & \longrightarrow (C_*(N)^{\otimes 2})^\vee \otimes_{(C_*(M_1^2)^{\otimes 2})^\vee}^{\mathbb{L}} (C_*(M_1^I)^{\otimes 2})^\vee \\
\Theta \otimes_\Theta \Theta \uparrow & & \uparrow \Theta \otimes_\Theta \Theta \\
C^*(N)^{\otimes 2} \otimes_{C^*(M_2^2)^{\otimes 2}}^{\mathbb{L}} C^*(M_2^I)^{\otimes 2} & \xrightarrow{1^{\otimes 2} \otimes_{(g^2)^* \otimes 2} (g^I)^* \otimes 2} & C^*(N)^{\otimes 2} \otimes_{C^*(M_1^2)^{\otimes 2}}^{\mathbb{L}} C^*(M_1^I)^{\otimes 2} \\
\top \uparrow & & \uparrow \top \\
(C^*(N) \otimes_{C^*(M_2^2)}^{\mathbb{L}} C^*(M_2^I))^{\otimes 2} & \xrightarrow{(1 \otimes_{(g^2)^*} (g^I)^*)^{\otimes 2}} & (C^*(N) \otimes_{C^*(M_1^2)}^{\mathbb{L}} C^*(M_1^I))^{\otimes 2}.
\end{array}$$

Thus the commutativity in the torsion functor yields (i). The assertion (ii) is shown by the same argument as in the proof of Proposition 9.2 (i).  $\square$

*Proof of Theorem 9.1.* With the same notation as in Propositions 9.3 and 9.4, we define a functor  $\mathbb{H}_*(L?M)$  by  $\mathbb{H}_*(L?M)(N) = \mathbb{H}_*(L_N M)$  and

$$\mathbb{H}_*(L?M)(f) = \widetilde{F}^! : \mathbb{H}_*(L_{N_2} M) \rightarrow \mathbb{H}_*(L_{N_1} M)$$

for a morphism  $f : N_1 \rightarrow N_2$  in **Poincaré** $_M$ . Proposition 2.7 implies that  $\mathbb{H}_*(L_N M)$  is a unital associative algebra over  $H_*(\Omega M)$ . In fact, the based map  $* \rightarrow M$  gives rise to an algebra map  $\mathbb{H}_*(L_N M) \rightarrow \mathbb{H}_*(L_M M) = H_{*+0}(\Omega M)$ ; see Proposition 9.3. It is readily seen that  $\mathbb{H}_*(L?M)(id_N) = id_{\mathbb{H}_*(L_N M)}$ . Moreover, the uniqueness of the shriek map enables us to deduce that  $\mathbb{H}_*(L?M)(fg) = \mathbb{H}_*(L?M)(g) \circ \mathbb{H}_*(L?M)(f)$ ; see Theorem 2.1. Then  $\mathbb{H}_*(L?M)$  is a well-defined functor.

The result on the naturality of the spectral sequence follows from Proposition 9.2 (ii) and the proof of Theorem 2.11, namely the construction of the spectral sequence converging to the relative loop homology.

We define a functor  $\mathbb{H}_*(L_N?)$  by  $\mathbb{H}_*(L_N?)(M) = \mathbb{H}_*(L_N M)$  and  $\mathbb{H}_*(L_N?)(g) = (\tilde{g})^\vee : \mathbb{H}_*(L_N M_1) \rightarrow \mathbb{H}_*(L_N M_2)$  for a morphism  $g : M_1 \rightarrow M_2$  in **Top** $_1^N$ . The well-definedness follows from Proposition 9.4.  $\square$

We end this section with computations of the relative loop product.

**Proposition 9.5.** *Let  $f : M \rightarrow K(\mathbb{Z}, 2) = BS^1$  be a map from a simply-connected Poincaré duality space  $M$ . Then  $\mathbb{H}_*(L_M K(\mathbb{Z}, 2); \mathbb{K}) \cong H^*(M; \mathbb{K}) \otimes \wedge(y)$  as an algebra, where  $\deg x \otimes y = -\deg x + 1$  for  $x \in H^*(M; \mathbb{K})$ .*

*Proof.* Let  $\{\mathbb{E}_r^{*,*}, d_r\}$  be the spectral sequence in Theorem 2.11 converging to the algebra  $\mathbb{H}_*(L_M K(\mathbb{Z}, 2); \mathbb{K})$ . Then it follows from [25, Proposition 2.4] that  $\mathbb{E}_2^{*,*} \cong H^*(M; \mathbb{K}) \otimes \wedge(y)$  as a bigraded algebra, where  $\text{bideg } y = (1, -2)$ . We see that

$\mathbb{E}_2^{p,*} = 0$  for  $* \geq 2$ . This yields that the spectral sequence collapses at the  $E_2$ -term and that  $xy - yx = 0$  and  $y^2 = 0$  in  $\mathbb{H}_*(L_M K(\mathbb{Z}, 2); \mathbb{K})$  for any  $x \in H^*(M)$ . Proposition 8.1 and Theorem 8.2 enable us to solve all extension problems. The answers are trivial. We thus have the result.  $\square$

By making use of functors  $\mathbb{H}_*(L?M)$  and  $\mathbb{H}_*(L_N?)$ , we compute the (relative) loop homology of a homogeneous space.

Let  $G$  be a simply-connected Lie group containing  $SU(2)$  as a subgroup and  $\pi : G \rightarrow G/SU(2)$  the projection. Suppose that the cohomology  $H^*(G; \mathbb{K})$  is isomorphic to an exterior algebra on generators with odd degree, say  $\wedge(V)$ . Moreover, we introduce the following condition (P):

The map  $i_* : H_3(SU(2); \mathbb{K}) \rightarrow H_3(G; \mathbb{K})$  induced by the inclusion  $i : SU(2) \rightarrow G$  is a monomorphism.

**Proposition 9.6.** *With the same assumption on a Lie group  $G$  as above, suppose further that the condition (P) holds. Then for some decomposition  $\mathbb{K}\{y_1\} \oplus \mathbb{K}\{y_2, \dots, y_l\}$  of  $V$ , one has a diagram*

$$\begin{array}{ccccc} \mathbb{H}_*(L(G/SU(2))) & \xleftarrow[\cong]{\quad} & \wedge(y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_2^*, \dots, \nu_l^*] \\ \rho := \mathbb{H}_*(L?(G/SU(2)))(\pi) \downarrow & & \\ \mathbb{H}_*(L_G(G/SU(2))) & \xleftarrow[\cong]{\quad} & \wedge(y'_1, y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_2^*, \dots, \nu_l^*] \\ \rho' := \mathbb{H}_*(L_G?)(\pi) \uparrow & & \\ \mathbb{H}_*(LG) & \xleftarrow[\cong]{\quad} & \wedge(y'_1, y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_1^*, \nu_2^*, \dots, \nu_l^*] \end{array}$$

in which the horizontal arrows are isomorphisms of algebras,  $\rho(y'_i) = y'_i$ ,  $\rho(\nu_i^*) = \nu_i^*$ ,  $\rho'(y'_i) = y'_i$ ,  $\rho'(\nu_1^*) = 0$  and  $\rho'(\nu_i^*) = \nu_i^*$  for  $i > 1$  up to isomorphism, where  $\deg \nu_i = \deg y_i - 1$  and  $\deg y'_i = -\deg y_i$ .

*Remark 9.7.* In general, for a simply-connected compact Lie group  $G$ , the homology  $H_3(G; \mathbb{Z})$  with coefficients in  $\mathbb{Z}$  is torsion free and its rank coincides with the number of simple factors of  $G$ ; see [40, Theorem 6.4.17] for example.

The result [40, Theorem 6.6.23] asserts that for any compact, simply-connected simple Lie group  $G$  there exists an inclusion  $SU(2) \rightarrow G$  such that the induced map  $i_* : H_3(SU(2); \mathbb{Z}) \rightarrow H_3(G; \mathbb{Z})$  is an isomorphism and then the condition (P) holds. On the other hand, as seen in [34, p. 767], there exist Lie groups containing  $SU(2)$  as a subgroup such that the induced map  $i_* : H_3(SU(2); \mathbb{Z}) \rightarrow H_3(G; \mathbb{Z})$  is multiplication by an integer greater than one. Thus we see that the condition (P) does not necessarily hold.

*Proof of Proposition 9.6.* Let  $\{E_r, d_r\}$  and  $\{E'_r, d'_r\}$  be the EMSS's associated with the fibrations  $G/SU(2) \rightarrow BSU(2) \rightarrow BG$  and  $G \rightarrow EG \rightarrow BG$ , respectively. We have a morphism of fibrations

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \pi \downarrow & & \downarrow & & \parallel \\ G/SU(2) & \longrightarrow & BSU(2) & \xrightarrow{Bi} & BG. \end{array}$$

This induces a morphism  $\{f_r\}$  of spectral sequences from  $\{E_r, d_r\}$  to  $\{E'_r, d'_r\}$ . Since the condition (P) holds, it follows that  $(Bi)^* : H^4(BG; \mathbb{K}) \rightarrow H^4(BSU(2); \mathbb{K})$  is an

epimorphism. Therefore there exists a decomposition  $\mathbb{K}\{y_1\} \oplus \mathbb{K}\{y_2, \dots, y_l\}$  of  $V$  such that, as bigraded algebras

$$E_2^{*,*} \cong \text{Tor}_{H^*(BG)}(\mathbb{K}, H^*(BSU(2))) \cong \wedge(y_2, \dots, y_l),$$

$$E'_2^{*,*} \cong \text{Tor}_{H^*(BG)}(\mathbb{K}, \mathbb{K}) \cong \wedge(y_1, y_2, \dots, y_l)$$

and  $f_2(y_i) = y_i$ . The algebra generators of the  $E_2$ -term in both the spectral sequences are in the second line. This implies that

$$H^*(G/SU(2)) \cong \wedge(y_2, \dots, y_l),$$

$$H^*(G) \cong \wedge(y_1, y_2, \dots, y_l)$$

and that  $\pi^*(y_i) = y_i$ . Let  $\{\mathbb{E}_r, d_r\}$ ,  $\{\mathbb{E}'_r, d'_r\}$  and  $\{\mathbb{E}''_r, d''_r\}$  be the spectral sequences converging to the loop homology  $\mathbb{H}_*(L(G/SU(2)))$ ,  $\mathbb{H}_*(LG)$  and  $\mathbb{H}_*(L_G(G/SU(2)))$  in Theorem 2.11, respectively. Theorem 9.1 (2) and the proof of [25, Proposition 2.4] yield the commutative diagram

$$\begin{array}{ccccc} \mathbb{E}_2 & \xleftarrow[\cong]{\quad} & HH^*(H^*(G/SU(2)), H^*(G/SU(2))) & \xlongequal{\quad} & \wedge(y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_2^*, \dots, \nu_l^*] \\ g_2 \downarrow & & \downarrow HH(1, \pi^*) & & \\ \mathbb{E}''_2 & \xleftarrow[\cong]{\quad} & HH^*(H^*(G/SU(2)), H^*(G)) & \xlongequal{\quad} & \wedge(y'_1, y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_2^*, \dots, \nu_l^*] \\ g'_2 \uparrow & & \uparrow HH(\pi^*, 1) & & \\ \mathbb{E}'_2 & \xleftarrow[\cong]{\quad} & HH^*(H^*(G), H^*(G)) & \xlongequal{\quad} & \wedge(y'_1, y'_2, \dots, y'_l) \otimes \mathbb{K}[\nu_1^*, \nu_2^*, \dots, \nu_l^*] \end{array}$$

for which  $g_2(y'_i) = y'_i$ ,  $g_2(\nu_i^*) = \nu_i^*$ ,  $g'_2(y'_i) = y'_i$ ,  $g'_2(\nu_1^*) = 0$  and  $g'_2(\nu_i^*) = \nu_i^*$  for  $i > 1$ , where  $\text{bideg } \nu_i = (1, -\deg y_i)$  and  $\text{bideg } y'_i = (0, \deg y_i)$ . It follows from Remark 8.7 that  $\{\mathbb{E}'_r, d'_r\}$  collapses at the  $E_2$ -term and that there is no extension problem on the  $E_\infty$ -term. This implies that  $\{\mathbb{E}''_r, d''_r\}$  collapses at the  $E_2$ -term and that  $y_i^2 = 0$  in  $\mathbb{H}_*(L_G(G/SU(2)))$ . Moreover, we see that there is no extension problem for commutativity and relations between generators since  $g'_2$  is an epimorphism.

Since the map  $g_2$  is a monomorphism, it follows that  $\{\mathbb{E}_r, d_r\}$  collapses at the  $E_2$ -term. Moreover, the same argument as in the proof of Theorem 8.6 with Theorem 8.2 yields that there is no extension problem on the  $E_\infty$ -term of  $\{\mathbb{E}_r, d_r\}$ . This completes the proof.  $\square$

## 10. PROOFS OF THEOREMS 2.14 AND 2.17

**Definition 10.1.** Let  $A$  be a (differential graded) algebra. Let  $M$  be a  $A$ -bimodule. Recall that we have a canonical map [37, p. 283]

$$\otimes_A : HH^*(A, M) \otimes HH^*(A, M) \rightarrow HH^*(A, M \otimes_A M).$$

(1) Let  $\bar{\mu}_M : M \otimes_A M \rightarrow M$  be a morphism of  $A$ -bimodules. Then the *cup product*  $\cup$  on  $HH^*(A, M)$  is the composite

$$HH^p(A, M) \otimes HH^q(A, M) \xrightarrow{\otimes_A} HH^{p+q}(A, M \otimes_A M) \xrightarrow{HH^{p+q}(A, \bar{\mu}_M)} HH^{p+q}(A, M).$$

(2) Let  $\varepsilon : Q \xrightarrow{\cong} M \otimes_A M$  be a  $A \otimes A^{op}$ -projective (semi-free) resolution of  $M \otimes_A M$ . Let  $\bar{\mu}_M \in \text{Ext}_{A \otimes A^{op}}(M \otimes_A M, M) = H(\text{Hom}_{A \otimes A^{op}}(Q, M))$ . Then the *generalized cup product*  $\cup$  on  $HH^*(A, M)$  is the composite

$$HH^*(A, M)^{\otimes 2} \xrightarrow{\otimes_A} HH^*(A, M \otimes_A M) \xrightarrow{HH^*(A, \varepsilon)^{-1}} HH^*(A, Q) \xrightarrow{HH^*(A, \bar{\mu}_M)} HH^*(A, M).$$

*Remark 10.2.* Let  $M$  be an associative (differential graded) algebra with unit  $1_M$ . Let  $h : A \rightarrow M$  be a morphism of (differential graded) algebras. Then

$$a \cdot m \star b := h(a)mh(b)$$

defines an  $A$ -bimodule structure on  $M$  such that the multiplication of  $M$ ,  $\mu_M : M \otimes M \rightarrow M$  induces a morphism of  $A$ -bimodules  $\bar{\mu}_M : M \otimes_A M \rightarrow M$ .

Conversely, let  $M$  be a  $A$ -bimodule equipped with an element  $1_M \in M$  and a morphism of  $A$ -bimodules  $\bar{\mu}_M : M \otimes_A M \rightarrow M$  such that  $\bar{\mu}_M \circ (\bar{\mu}_M \otimes_A 1) = \bar{\mu}_M \circ (1 \otimes_A \bar{\mu}_M)$  and such that the two maps  $m \mapsto \bar{\mu}_M(m \otimes_A 1)$  and  $m \mapsto \bar{\mu}_M(1 \otimes_A m)$  coincide with the identity map on  $M$ . Then the map  $h : A \rightarrow M$  defined by  $h(a) := a \cdot 1_M$  is a morphism of algebras.

The following lemma gives an interesting decomposition of the cup product of the Hochschild cohomology of a commutative (possible differential graded) algebra.

**Lemma 10.3.** *Let  $A$  be a commutative (differential graded) algebra. Let  $M$  be a  $A$ -module. Let  $B$  be an  $A^{\otimes 2}$ -module. Let  $\mu : A^{\otimes 2} \rightarrow A$  denote the multiplication of  $A$ . Let  $\eta : \mathbb{K} \rightarrow A$  be the unit of  $A$ . Let  $q : B \otimes B \rightarrow B \otimes_A B$  be the quotient map. Then*

- (1)  $\text{Tor}_*^{1 \otimes \mu \otimes 1}(1, \mu) : \text{Tor}_*^{A^{\otimes 4}}(M, A \otimes A) \xrightarrow{\cong} \text{Tor}_*^{A^{\otimes 3}}(M, A)$  is an isomorphism,
- (2)  $\text{Hom}_{1 \otimes \mu \otimes 1}(q, 1) : \text{Hom}_{A^{\otimes 3}}(B \otimes_A B, M) \xrightarrow{\cong} \text{Hom}_{A^{\otimes 4}}(B \otimes B, M)$  is an isomorphism and
- (3)  $\text{Ext}_{1 \otimes \mu \otimes 1}^*(q, 1) : \text{Ext}_{A^{\otimes 3}}^*(B \otimes_A B, M) \xrightarrow{\cong} \text{Ext}_{A^{\otimes 4}}^*(B \otimes B, M)$  is also an isomorphism.

(4) Let  $\mu_M \in \text{Hom}_{A^{\otimes 4}}(M^{\otimes 2}, M)$ . Then  $\mu_M$  induced a quotient map  $\bar{\mu}_M : M \otimes_A M \rightarrow M$  and the cup product  $\cup$  of the Hochschild cohomology of  $A$  with coefficients in  $M$ ,  $\text{HH}^*(A, M) = \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu}$  is given by the following commutative diagram

$$\begin{array}{ccc} \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} \otimes \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} & \xrightarrow{\otimes} & \text{Ext}_{A^{\otimes 4}}^*(A^{\otimes 2}, M^{\otimes 2})_{\mu^{\otimes 2}, \mu^{\otimes 2}} \\ \cup \downarrow & & \downarrow \text{Ext}_1^*(1, \mu_M) \\ & & \text{Ext}_{A^{\otimes 4}}^*(A^{\otimes 2}, M)_{\mu^{\otimes 2}, \mu \circ \mu^{\otimes 2}} \\ & & \cong \downarrow \text{Ext}_{1 \otimes \mu \otimes 1}^*(\mu, 1)^{-1} \\ \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} & \xleftarrow{\text{Ext}_{1 \otimes \eta \otimes 1}^*(1, 1)} & \text{Ext}_{A^{\otimes 3}}^*(A, M)_{\mu \circ (\mu \otimes 1), \mu \circ (\mu \otimes 1)} \end{array}$$

(5) Let  $\varepsilon : R \xrightarrow{\sim} M \otimes M$  be a  $A^{\otimes 4}$ -projective (semi-free) resolution of  $M \otimes M$ . Let  $\mu_M \in \text{Ext}_{A^{\otimes 4}}(M^{\otimes 2}, M) = H(\text{Hom}_{A^{\otimes 4}}(R, M))$ . Let  $\bar{\mu}_M$  be  $\text{Ext}_{1 \otimes \mu \otimes 1}^*(q, 1)^{-1}(\mu_M)$ . Then the generalized cup product  $\cup$  of the Hochschild cohomology of  $A$  with coefficients in  $M$ ,  $\text{HH}^*(A, M) = \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu}$  is given by the following commutative diagram

$$\begin{array}{ccc} \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} \otimes \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} & \xrightarrow{\otimes} & \text{Ext}_{A^{\otimes 4}}^*(A^{\otimes 2}, M^{\otimes 2})_{\mu^{\otimes 2}, \mu^{\otimes 2}} \\ \cup \downarrow & & \cong \downarrow (\text{Ext}_1^*(1, \varepsilon))^{-1} \\ & & \text{Ext}_{A^{\otimes 4}}^*(A^{\otimes 2}, R) \\ & & \downarrow \text{Ext}_1^*(1, \mu_M) \\ & & \text{Ext}_{A^{\otimes 4}}^*(A^{\otimes 2}, M)_{\mu^{\otimes 2}, \mu \circ \mu^{\otimes 2}} \\ & & \cong \downarrow \text{Ext}_{1 \otimes \mu \otimes 1}^*(\mu, 1)^{-1} \\ \text{Ext}_{A^{\otimes 2}}^*(A, M)_{\mu, \mu} & \xleftarrow{\text{Ext}_{1 \otimes \eta \otimes 1}^*(1, 1)} & \text{Ext}_{A^{\otimes 3}}^*(A, M)_{\mu \circ (\mu \otimes 1), \mu \circ (\mu \otimes 1)} \end{array}$$

*Proof.* (1) Consider the bar resolution  $\xi : B(A, A, A) \xrightarrow{\sim} A$  of  $A$ . Since the complex  $B(A, A, A)$  is a semifree  $A$ -module, it follows from [13, Theorem 6.1] that  $\xi \otimes_A \xi : B(A, A, A) \otimes_A B(A, A, A) \rightarrow A \otimes_A A = A$  is a quasi-isomorphism and hence it is a projective resolution of  $A$  as a  $A^{\otimes 3}$ -module. We moreover have a commutative diagram

$$\begin{array}{ccc} B(A, A, A) \otimes B(A, A, A) & \xrightarrow{\xi \otimes \xi} & A \otimes A \\ q \downarrow & & \downarrow \mu \\ B(A, A, A) \otimes_A B(A, A, A) & \xrightarrow{\xi \otimes_A \xi} & A \end{array}$$

in which  $q$  is the natural projection and the first row is a projective resolution of  $A \otimes A$  as a  $A^{\otimes 4}$ -module. It is immediate that  $q$  is a morphism of  $A^{\otimes 4}$ -modules with respect to the morphism of algebras  $1 \otimes \mu \otimes 1 : A^{\otimes 4} \rightarrow A^{\otimes 3}$ . Then  $\text{Tor}_{1 \otimes \mu \otimes 1}(1, \mu)$  is induced by the map

$$1 \otimes q : M \otimes_{A^{\otimes 4}} B(A, A, A) \otimes B(A, A, A) \rightarrow M \otimes_{A^{\otimes 3}} B(A, A, A) \otimes_A B(A, A, A).$$

Since  $A$  is a commutative, it follows that both the source and target of  $1 \otimes q$  are isomorphic to  $W := M \otimes B(\mathbb{K}, A, \mathbb{K}) \otimes B(\mathbb{K}, A, \mathbb{K})$  as a vector space. As a linear map,  $1 \otimes q$  coincides with the identity map on  $W$  up to isomorphism.

(2) By the universal property of the quotient map  $q : B \otimes B \twoheadrightarrow B \otimes_A B$ ,  $\text{Hom}_{1 \otimes \mu \otimes 1}(q, 1)$  is an isomorphism.

(3) Let  $\varepsilon : P \xrightarrow{\sim} B$  be an  $A^{\otimes 2}$ -projective (semifree) resolution of  $B$ . We have a commutative square of  $A^{\otimes 4}$ -modules

$$\begin{array}{ccc} P \otimes P & \xrightarrow[\simeq]{\varepsilon \otimes \varepsilon} & B \otimes B \\ q' \downarrow & & \downarrow q \\ P \otimes_A P & \xrightarrow[\simeq]{\varepsilon \otimes_A \varepsilon} & B \otimes_A B \end{array}$$

Therefore  $\text{Ext}_{1 \otimes \mu \otimes 1}^*(q, 1)$  is induced by  $\text{Hom}_{1 \otimes \mu \otimes 1}(q', 1)$  which is an isomorphism by (2).

(4) Let  $A$  be any algebra and  $M$  be any  $A$ -bimodule. Let  $\xi : \mathbb{B} \xrightarrow{\sim} A$  an  $A \otimes A^{op}$ -projective (semi-free) resolution (for example the double bar resolution). Let  $c : \mathbb{B} \rightarrow \mathbb{B} \otimes_A \mathbb{B}$  be a morphism of  $A$ -bimodules such that the diagram of  $A$ -bimodules

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\xi} & A \\ \downarrow c & & \downarrow \cong \\ \mathbb{B} \otimes_A \mathbb{B} & \xrightarrow{\xi \otimes_A \xi} & A \otimes_A A \end{array}$$

is homotopy commutative. The cup product of  $f$  and  $g \in \text{Hom}_{A \otimes A^{op}}(\mathbb{B}, M)$  is the composite  $\bar{\mu}_M \circ (f \otimes_A g) \circ c \in \text{Hom}_{A \otimes A^{op}}(\mathbb{B}, M)$  [43, p. 134].

Suppose now that  $A$  is commutative and that the  $A$ -bimodule structure on  $M$  comes from the multiplication  $\mu$  of  $A$  and an  $A$ -module structure on  $M$ . The following diagram of complexes gives two different decompositions of the cup product

on  $\text{Hom}_{A \otimes A^{op}}(\mathbb{B}, M)$ .

$$\begin{array}{ccc}
\text{Hom}_{A \otimes 2}(\mathbb{B}, M) \otimes \text{Hom}_{A \otimes 2}(\mathbb{B}, M) & \xrightarrow{\otimes} & \text{Hom}_{A \otimes 4}(\mathbb{B} \otimes \mathbb{B}, M \otimes M) \\
\downarrow \otimes_A & & \downarrow \text{Hom}_1(1, \mu_M) \\
\text{Hom}_{A \otimes 3}(\mathbb{B} \otimes_A \mathbb{B}, M \otimes_A M) & \xrightarrow{\text{Hom}_1(1, \bar{\mu}_M)} & \text{Hom}_{A \otimes 4}(\mathbb{B} \otimes \mathbb{B}, M) \\
& \searrow & \cong \downarrow \text{Hom}_{1 \otimes \mu \otimes 1}(q, 1) \\
\text{Hom}_{A \otimes A^{op}}(\mathbb{B} \otimes_A \mathbb{B}, M) & \xleftarrow{\text{Hom}_{1 \otimes \eta \otimes 1}(1, 1)} & \text{Hom}_{A \otimes 3}(\mathbb{B} \otimes_A \mathbb{B}, M) \\
\downarrow \text{Hom}_1(c, 1) & & \\
\text{Hom}_{A \otimes A^{op}}(\mathbb{B}, M) & &
\end{array}$$

(5) Let  $\varepsilon : P \xrightarrow{\sim} M$  be a surjective  $A \otimes A^{op}$ -projective (semifree) resolution of  $M$ . Then  $\mu_M$  can be considered as an element of  $\text{Hom}_{A \otimes 4}(P \otimes P, M)$ . By lifting, there exists  $\mu_P \in \text{Hom}_{A \otimes 4}(P \otimes P, P)$  such that  $\varepsilon \circ \mu_P = \mu_M$ . By 2), there exists  $\bar{\mu}_P$  such that  $\bar{\mu}_P \circ q = \mu_P$ . We can take  $\bar{\mu}_M = \varepsilon \circ \bar{\mu}_P$ .

It is now easy to check that the isomorphism  $HH^*(A, \varepsilon) : HH^*(A, P) \xrightarrow{\cong} HH^*(A, M)$  transports the cup product on  $HH^*(A, P)$  defined using  $\bar{\mu}_P$  to the generalized cup product on  $HH^*(A, M)$  defined using  $\bar{\mu}_M$ . We now check that the isomorphism  $HH^*(A, \varepsilon) : HH^*(A, P) \xrightarrow{\cong} HH^*(A, M)$  transports the composite

$$\begin{array}{ccc}
\text{Ext}_{A \otimes 2}^*(A, P)_{\mu, \mu} \otimes \text{Ext}_{A \otimes 2}^*(A, P)_{\mu, \mu} & \xrightarrow{\otimes} & \text{Ext}_{A \otimes 4}^*(A^{\otimes 2}, P^{\otimes 2})_{\mu^{\otimes 2}, \mu^{\otimes 2}} \\
& & \downarrow \text{Ext}_1^*(1, \mu_P) \\
& & \text{Ext}_{A \otimes 4}^*(A^{\otimes 2}, P)_{\mu^{\otimes 2}, \mu \circ \mu^{\otimes 2}} \\
& & \cong \downarrow \text{Ext}_{1 \otimes \mu \otimes 1}^*(\mu, 1)^{-1} \\
\text{Ext}_{A \otimes 2}^*(A, P)_{\mu, \mu} & \xleftarrow{\text{Ext}_{1 \otimes \eta \otimes 1}^*(1, 1)} & \text{Ext}_{A \otimes 3}^*(A, P)_{\mu \circ (\mu \otimes 1), \mu \circ (\mu \otimes 1)}
\end{array}$$

into the composite

$$\begin{array}{ccc}
\text{Ext}_{A \otimes 2}^*(A, M)_{\mu, \mu} \otimes \text{Ext}_{A \otimes 2}^*(A, M)_{\mu, \mu} & \xrightarrow{\otimes} & \text{Ext}_{A \otimes 4}^*(A^{\otimes 2}, M^{\otimes 2})_{\mu^{\otimes 2}, \mu^{\otimes 2}} \\
& & \cong \downarrow (\text{Ext}_1^*(1, \varepsilon \otimes \varepsilon))^{-1} \\
& & \text{Ext}_{A \otimes 4}^*(A^{\otimes 2}, P \otimes P) \\
& & \downarrow \text{Ext}_1^*(1, \mu_M) \\
& & \text{Ext}_{A \otimes 4}^*(A^{\otimes 2}, M)_{\mu^{\otimes 2}, \mu \circ \mu^{\otimes 2}} \\
& & \cong \downarrow \text{Ext}_{1 \otimes \mu \otimes 1}^*(\mu, 1)^{-1} \\
\text{Ext}_{A \otimes 2}^*(A, M)_{\mu, \mu} & \xleftarrow{\text{Ext}_{1 \otimes \eta \otimes 1}^*(1, 1)} & \text{Ext}_{A \otimes 3}^*(A, M)_{\mu \circ (\mu \otimes 1), \mu \circ (\mu \otimes 1)}
\end{array}$$

By applying (4) to  $\mu_P$ , we have proved (5).  $\square$

**Theorem 10.4.** (Compare with [15, Theorem 12]) Let  $B$  be a simply-connected commutative Gorenstein cochain algebra of dimension  $m$  such that  $\forall i \in \mathbb{N}$ ,  $H^i(B)$  is finite dimensional. Then

$$\text{Ext}_{B \otimes B}^{*+m}(B, B \otimes B) \cong H^*(B).$$

*Proof.* The proof of [15, Theorem 12] for the strongly homotopy commutative algebra  $C^*(X)$  obviously works in the case of a commutative algebra  $B$ .  $\square$

*Remark 10.5.* In [1, Theorem 2.1 i) iv)] Avramov and Iyengar have shown a related result in the non graded case: Let  $S$  be a commutative algebra over a field  $\mathbb{K}$ , which is the quotient of a polynomial algebra  $\mathbb{K}[x_1, \dots, x_d]$  or more generally which

is the quotient of a localization of  $\mathbb{K}[x_1, \dots, x_d]$ . Then  $S$  is Gorenstein if and only if the graded  $S$ -module  $\text{Ext}_{S \otimes S}^*(S, S \otimes S)$  is projective of rank 1.

*Example 10.6.* (The generalized cup product of a Gorenstein algebra) Let  $A \rightarrow B$  be a morphism of commutative differential graded algebras where  $B$  satisfies the hypotheses of Theorem 10.4. Let  $\Delta_B : B \rightarrow B \otimes B$  be a generator of  $\text{Ext}_{B \otimes B}^m(B, B \otimes B) \cong \mathbb{K}$ . By taking duals, we obtain the following element of  $\text{Ext}_{A \otimes A}^m(B^\vee \otimes B^\vee, B^\vee)$ :

$$(\Delta_B)^\vee : B^\vee \otimes B^\vee \rightarrow (B \otimes B)^\vee \rightarrow B^\vee.$$

By 3) of Lemma 10.3,  $(\Delta_B)^\vee$  induces an element  $\bar{\mu}_{B^\vee} \in \text{Ext}_{A \otimes A}^m(B^\vee \otimes_A B^\vee, B^\vee)$ . Therefore, by ii) of definition 10.1, we have a generalized cup product

$$HH^p(A, B^\vee) \otimes HH^q(A, B^\vee) \xrightarrow{\cup} HH^{p+q+m}(A, B^\vee).$$

*Proof of Theorem 2.17.* Step 1: The polynomial differential functor  $A(X)$  extends to a functor  $A(X, Y)$  for pairs of spaces  $Y \subset X$ . The two natural short exact sequences [13, p. 124]

$$0 \rightarrow A(X, Y) \rightarrow A(X) \rightarrow A(Y) \rightarrow 0$$

and

$$0 \rightarrow C^*(X, Y) \rightarrow C^*(X) \rightarrow C^*(Y) \rightarrow 0$$

are naturally weakly equivalent [13, p. 127-8]. Therefore all the results of Felix and Thomas given in [15] with the singular cochains algebra  $C^*(X; \mathbb{Q})$  are valid with  $A(X)$  (For example, the description of the shriek map of an embedding  $N \hookrightarrow M$  at the level of singular cochains given p. 419 of [15]). In particular, our Theorem 2.3 is valid when we replace  $C^*(X)$  by  $A(X)$ . (Note also that a proof similar to the proof of Theorems 11.3 or 11.6 shows that the dual of the loop product on  $A(L_N M)$  is isomorphic to the dual of the loop product defined on  $C^*(L_N M)$ .) This means the following: Let  $\Delta^!$  be a generator of  $\text{Ext}_{A(N^2)}^m(A(N), A(N^2)) \cong \mathbb{Q}$  given by Theorem 12 of [15]. Then  $\text{Tor}^1(1, \sigma^*) \circ EM^{-1}$  is an isomorphism of algebras between the dual of the loop product  $Dlp$  on  $H^*(A(L_N M))$  and the coproduct defined by the composite on the left column of the following diagram.

Step 2: We have chosen  $\Delta^!$  and  $\Delta_{A(N)}$  such that the composite  $\varphi \circ \Delta_{A(N)}$  is equal to  $\Delta^!$  in the derived category of  $A(N)^{\otimes 2}$ -modules. Therefore the following diagram commutes.

$$\begin{array}{ccc} \text{Tor}_*^{A(M^2)}(A(N), A(M)) & \xleftarrow[\cong]{\text{Tor}^{\varphi(1,1)}} & \text{Tor}_*^{A(M)^{\otimes 2}}(A(N), A(M)) \\ \text{Tor}^{A(p_{13})(1,1)} \downarrow & & \text{Tor}^{1 \otimes \eta \otimes 1}(1,1) \downarrow \\ \text{Tor}_*^{A(M^3)}(A(N), A(M)) & \xleftarrow[\cong]{\text{Tor}^{\varphi \circ (1 \otimes \varphi)}(1,1)} & \text{Tor}_*^{A(M)^{\otimes 3}}(A(N), A(M)) \\ \text{Tor}^{A(1 \times \Delta \times 1)}(1, A(\Delta)) \uparrow \cong & & \text{Tor}^{1 \otimes \mu \otimes 1}(1, \mu) \uparrow \cong \\ \text{Tor}_*^{A(M^4)}(A(N), A(M^2)) & \xleftarrow[\cong]{\text{Tor}^{\varphi(1, \varphi)}} & \text{Tor}_*^{A(M)^{\otimes 4}}(A(N), A(M)^{\otimes 2}) \\ \text{Tor}^1(\Delta^!, 1) \downarrow & & \text{Tor}^1(\Delta_{A(N)}, 1) \downarrow \\ \text{Tor}_*^{A(M^4)}(A(N^2), A(M^2)) & \xleftarrow[\cong]{\text{Tor}^{\varphi(\varphi, \varphi)}} & \text{Tor}_*^{A(M)^{\otimes 4}}(A(N)^{\otimes 2}, A(M)^{\otimes 2}) \\ \cong \downarrow & & \cong \downarrow \\ \left( \text{Tor}_*^{A(M^2)}(A(N), A(M)) \right)^{\otimes 2} & \xleftarrow[\cong]{(\text{Tor}^{\varphi(1,1)})^{\otimes 2}} & \left( \text{Tor}_*^{A(M)^{\otimes 2}}(A(N), A(M)) \right)^{\otimes 2} \end{array}$$

Step 3: Dualizing and using the natural isomorphism

$$\mathrm{Ext}_B^*(Q, P^\vee) \xrightarrow{\cong} \mathrm{Tor}_B^*(P, Q)^\vee$$

for any differential graded algebra  $B$ , right  $B$ -module  $P$  and left  $B$ -module  $Q$ , we see that the dual of  $\Phi$  is an isomorphism of algebras with respect to the loop product and to the long composite given by the diagram of (5) of Lemma 10.3 when  $A := A(M)$ ,  $M := A(N)^\vee$  and  $\mu_M := (\Delta_{A(N)})^\vee$ .

Step 4: We apply part 5) of Lemma 10.3 to see that this long composite coincides with the generalized cup product of the Gorenstein algebra  $B := A(N)$ . The same argument as in the proof of Theorem 5.3 completes the proof.  $\square$

*Proof of Theorem 2.14.* Let  $\varepsilon : \mathbb{B} \xrightarrow{\sim} C^*(M)$  be a right  $C^*(M^2)$ -semifree resolution of  $C^*(M)$ . As explained in p. 419 of [15],  $\Delta^!$  fits into the following homotopy commutative diagram of right  $C^*(M^2)$ -modules.

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\Delta^!} & C^{*+m}(M^2) \\ \varepsilon \downarrow \simeq & & \simeq \downarrow \cap_{M^2} \\ C^*(M) & \xrightarrow[\simeq]{\cap_M} & C_*(M) \xrightarrow{\Delta_*} C_*(M^2) \end{array}$$

Therefore, by applying the functor  $\mathrm{Tor}_{C^*(M^4)}^*(-, C^*(M^2))$ , we obtain the commutative square

$$\begin{array}{ccc} \mathrm{Tor}_{C^*(M^4)}^*(C^*(M), C^*(M^2)) & \xrightarrow{\mathrm{Tor}_1(\cap_M, 1)} & \mathrm{Tor}_{C^*(M^4)}^*(C_*(M), C^*(M^2)) \\ \mathrm{Tor}_1(\Delta^!, 1) \downarrow & & \downarrow \mathrm{Tor}_1(\Delta_*, 1) \\ \mathrm{Tor}_{C^*(M^4)}^*(C^*(M^2), C^*(M^2)) & \xrightarrow{\mathrm{Tor}_1(\cap_{M^2}, 1)} & \mathrm{Tor}_{C^*(M^4)}^*(C_*(M^2), C^*(M^2)). \end{array}$$

Therefore using Theorem 2.3,  $\Phi$  is an isomorphism of coalgebras with respect to the dual of the loop product and to the following composite

$$\begin{array}{ccc} \mathrm{Tor}_{C^*(M^2)}^*(C_*M, C^*M) & \xrightarrow{\mathrm{Tor}_{p_{13}^*}(1, 1)} & \mathrm{Tor}_{C^*(M^3)}^*(C_*M, C^*M) \\ & & \cong \uparrow \mathrm{Tor}_{(1 \times \Delta \times 1)^*}(1, \Delta^*) \\ \mathrm{Tor}_{C^*(M^4)}^*(C_*(M^2), C^*(M^2)) & \xleftarrow{\mathrm{Tor}_1(\Delta_*, 1)} & \mathrm{Tor}_{C^*(M^4)}^*(C_*(M), C^*(M^2)) \\ & \cong \uparrow \mathrm{Tor}_1(EZ, 1) & \\ \mathrm{Tor}_{C^*(M^4)}^*(C_*(M) \otimes^2, C^*(M^2)) & \xrightarrow[\mathrm{Tor}_{EZ^\vee}(1, EZ^\vee)]{\cong} & \mathrm{Tor}_{(C_*(M^2) \otimes^2)^\vee}^*(C_*(M) \otimes^2, (C_*(M) \otimes^2)^\vee) \\ & & \cong \uparrow \mathrm{Tor}_\gamma(1, \gamma) \\ \mathrm{Tor}_{C^*(M^2)}^*(C_*(M), C^*(M))^{\otimes 2} & \xrightarrow[\top]{\cong} & \mathrm{Tor}_{C^*(M^2) \otimes^2}^*(C_*(M) \otimes^2, C^*(M) \otimes^2). \end{array}$$

Dualizing and using the natural isomorphism

$$\mathrm{Ext}_B^*(Q, P^\vee) \xrightarrow{\cong} \mathrm{Tor}_B^*(P, Q)^\vee$$

for any differential graded algebra  $B$ , right  $B$ -module  $P$  and left  $B$ -module  $Q$ , we see that the dual of  $\Phi$  is an isomorphism of algebras with respect to the loop product and to the multiplication defined in Theorem 2.14.  $\square$

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## 11. APPENDIX: PROPERTIES OF SHRIEK MAPS

In this section, we extend the definitions and properties of shriek maps on Gorenstein spaces given in [15].

**Definition 11.1.** A pull-back diagram,

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{f} & M \end{array}$$

satisfies Hypothesis (H) (Compare with the hypothesis (H) described in [15, page 418]) if  $p : E \rightarrow M$  is a fibration, for any  $n \in \mathbb{N}$ ,  $H^n(E)$  is of finite dimension and

$$\begin{cases} N \text{ is an oriented Poincaré duality space of dimension } n, \\ M \text{ is a 1-connected oriented Poincaré duality space of dimension } m, \end{cases}$$

or  $f : B^r \rightarrow B^t$  is the product of diagonal maps  $B \rightarrow B^{n_i}$ , the identity map of  $B$ , the inclusion  $\eta : * \rightarrow B$  for a simply-connected  $\mathbb{K}$ -Gorenstein space  $B$ .

Let  $n$  be the dimension of  $N$  or  $r$  times the dimension of  $B$ . Let  $m$  be the dimension of  $M$  or  $t$  times the dimension of  $B$ . It follows from [15, Lemma 1 and Corollary p. 448] that  $H^q(N) \cong \text{Ext}_{C^*(M)}^{q+m-n}(C^*(N), C^*(M))$ . By definition, a shriek map  $f^!$  for  $f$  is a generator of  $\text{Ext}_{C^*(M)}^{\leq m-n}(C^*(N), C^*(M))$ . Moreover, there exists a unique element  $g^! \in \text{Ext}_{C^*(E)}^{m-n}(C^*(X), C^*(E))$  such that  $g^! \circ C^*(q) = C^*(p) \circ f^!$  in the derived category of  $C^*(M)$ -modules; see Theorem 2.1.

Here we have extended the definitions of shriek maps due to Felix and Thomas in order to include the following example and the case  $(\Delta \times 1)^!$  that we use in the proof of Proposition 2.7.

*Example 11.2.* (Compare with [15, p. 419-420] where  $M$  is a Poincaré duality space) Let  $F \xrightarrow{\tilde{\eta}} E \xrightarrow{p} M$  be a fibration over a simply-connected Gorenstein space  $M$  with generator  $\eta^! = \omega_M \in \text{Ext}_{C^*(M)}^m(\mathbb{K}, C^*(M))$ . By definition,  $H(\tilde{\eta}^!) : H^*(F) \rightarrow H^{*+m}(E)$  is the dual to the *intersection morphism*.

Let  $G$  be a connected Lie group. Then its classifying space  $BG$  is an example of Gorenstein space of negative dimension. Let  $F$  be a  $G$ -space. It is not difficult to see that our intersection morphism of  $F \rightarrow F \times_G EG \rightarrow BG$  coincides with the integration along the fibre of the principal  $G$ -fibration  $G \rightarrow F \times EG \rightarrow F \times_G EG$  for an appropriate choice of the generator  $\eta^!$ ; see the proof of [15, Theorem 6].

Suppose now that  $F \xrightarrow{\tilde{\eta}} E \xrightarrow{p} M$  is a monoidal fibration. With the properties of shriek maps given in this section, generalizing [15, Theorem 10] (See also [19, Proposition 10]) in the Gorenstein case, one can show that the intersection morphism  $H(\tilde{\eta}^!) : H_{*+m}(E) \rightarrow H_*(F)$  is multiplicative if in the derived category of  $C^*(M \times M)$ -modules

$$(11.1) \quad \Delta^! \circ \omega_M = \omega_M \times \omega_M.$$

The generator  $\Delta^! \in \text{Ext}_{C^*(M^2)}^m(C^*(M), C^*(M^2))$  is defined up to a multiplication by a scalar. If we could prove that  $\Delta^! \circ \omega_M$  is always not zero, we would have an

unique choice for  $\Delta^!$  satisfying (11.1). Then we would have solved the “up to a constant problem” mentioned in [15, Q1 p. 423].

We now describe a generalized version of [15, Theorem 3]. We consider the following commutative diagram.

$$\begin{array}{ccccc}
 & X & \xrightarrow{g} & E & \\
 X' \xleftarrow{v} & \downarrow g' & & \swarrow k & \downarrow p \\
 & E' & \xrightarrow{k} & M & \\
 q \downarrow & \xrightarrow{q'} & N & \xrightarrow{f} & M' \\
 N' \xleftarrow{u} & \downarrow f' & & \swarrow h & \\
 & M' & & &
 \end{array}$$

in which the back and the front squares satisfies Hypothesis (H).

**Theorem 11.3.** (*Compare with [15, Theorem 3]*) *With the above notations, suppose that  $m' - n' = m - n$ .*

(1) *If  $h$  is a homotopy equivalence then in the derived category of  $C^*(M')$ -modules,  $f^! \circ C^*(u) = \varepsilon C^*(h) \circ f'^!$ , where  $\varepsilon \in \mathbb{K}$ .*

(2) *If in the derived category of  $C^*(M')$ -modules,  $f^! \circ C^*(u) = \varepsilon C^*(h) \circ f'^!$  then in the derived category of  $C^*(E')$ -modules,  $g^! \circ C^*(v) = \varepsilon C^*(k) \circ g'^!$ . In particular,*

$$H^*(g^!) \circ H^*(v) = \varepsilon H^*(k) \circ H^*(g'^!).$$

*Remark 11.4.* a) In (1), if  $N'$  and  $M'$  are oriented Poincaré duality spaces, the constant  $\varepsilon$  is given by

$$H^n(f^!) \circ H^n(u)(\omega_{N'}) = \varepsilon H^{m'}(h) \circ H^{n'}(f'^!)(\omega_{N'}).$$

In fact, this is extractd from the uniqueness of the shriek map described in [15, Lemma 1].

b) In [15, Theorem 3], it is not useful that  $v$  and  $k$  are homotopy equivalence. But in [15, Theorem 3], the homotopy equivalences  $u$  and  $h$  should be orientation preserving in order to deduce  $\varepsilon = 1$ .

c) If the bottom square is the pull-back along a smooth embedding  $f'$  of compact oriented manifolds and a smooth map  $h$  transverse to  $N'$ . Then by [36, Proposition 4.2],  $f^! \circ C^*(u) = C^*(h) \circ f'^!$  and  $H^*(g^!) \circ H^*(v) = H^*(k) \circ H^*(g'^!)$ .

*Proof of Theorem 11.3.* The proofs of (1) and (2) follow from the proof of [15, Theorem 3]. But we review this proof, in order to explain that Theorem 11.3 is valid in the Gorenstein case and that we don't need to assume as in [15, Theorem 3] that  $u$ ,  $k$  and  $v$  are homotopy equivalence.

(1) Since  $h$  is a homotopy equivalence,

$$\mathrm{Ext}_{C^*(M')}^*(C^*(N'), C^*(h)) :$$

$$\mathrm{Ext}_{C^*(M')}^*(C^*(N'), C^*(M')) \rightarrow \mathrm{Ext}_{C^*(M')}^*(C^*(N'), C^*(M))$$

is an isomorphism. By definition [15, Theorem 1 and p. 449], the shriek map  $f'^!$  is a generator of  $\mathrm{Ext}_{C^*(M')}^{m'-n'}(C^*(N'), C^*(M')) \cong \mathbb{K}$ . Then  $C^*(h) \circ f'^!$  is a generator of  $\mathrm{Ext}_{C^*(M')}^{m'-n'}(C^*(N'), C^*(M))$ . So since  $f^! \circ C^*(u)$  is in  $\mathrm{Ext}_{C^*(M')}^{m-n}(C^*(N'), C^*(M))$ , we have (1).

(2) Let  $P$  be any  $C^*(E')$ -module. Since  $X'$  is a pull-back, a straightforward generalization of [15, Theorem 2] shows that

$$\mathrm{Ext}_{C^*(p')}^*(C^*(q'), P) : \mathrm{Ext}_{C^*(E')}^*(C^*(X'), P) \rightarrow \mathrm{Ext}_{C^*(M')}^*(C^*(N'), P)$$

is an isomorphism. Take  $P := C^*(E)$ . Consider the following cube in the derived category of  $C^*(M')$ -modules.

$$\begin{array}{ccccc} & C^*(X) & \xrightarrow{g^!} & C^*(E) & \\ C^*(v) \swarrow \curvearrowright C^*(q) & \uparrow g'^! & & \nearrow C^*(k) & \\ C^*(X') & \xrightarrow{\quad} & C^*(E') & \xrightarrow{\quad} & C^*(M) \\ \uparrow C^*(q') & \uparrow C^*(u) & \uparrow f^! & \uparrow C^*(p') & \uparrow C^*(h) \\ C^*(N') & \xrightarrow{f'^!} & C^*(M') & \xrightarrow{\quad} & \end{array}$$

Since in  $\mathrm{Ext}_{C^*(M')}^*(C^*(N'), C^*(E))$ , the elements  $g^! \circ C^*(v) \circ C^*(q')$  and  $\varepsilon C^*(k) \circ g'^! \circ C^*(q')$  are equal, the assertion (2) follows.  $\square$

When  $u$  and  $h$  are the identity maps, Theorem 11.3 gives [15, Theorem 4] (Compare with [19, Lemma 4]) and the following variant for Gorenstein spaces:

**Theorem 11.5.** (*Naturality of shriek maps with respect to pull-backs*) Consider the two pull-back squares

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ v \downarrow & & \downarrow k \\ X' & \xrightarrow{g'} & E' \\ q' \downarrow & & \downarrow p' \\ B^r & \xrightarrow{\Delta} & B^t \end{array}$$

where  $\Delta : B^r \rightarrow B^t$  is the product of diagonal maps of a simply-connected  $\mathbb{K}$ -Gorenstein space  $B$  and  $p'$  and  $p' \circ k$  are two fibrations. Then in the derived category of  $C^*(E')$ -modules,

$$g^! \circ C^*(v) = C^*(k) \circ g'^!.$$

**Theorem 11.6.** (*Products of shriek maps*) Let

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{f} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} X' & \xrightarrow{g'} & E' \\ q' \downarrow & & \downarrow p' \\ N' & \xrightarrow{f'} & M' \end{array}$$

be two pull-back diagrams satisfying Hypothesis (H). Let

$$EZ^\vee : C^*(M \times M') \xrightarrow{\sim} (C_*(M) \otimes C_*(M'))^\vee$$

be the quasi-isomorphism of algebras dual to the Eilenberg-Zilber morphism. Let

$$\Theta : C^*(M) \otimes C^*(M') \xrightarrow{\sim} (C_*(M) \otimes C_*(M'))^\vee$$

be the quasi-isomorphism of algebras sending the tensor product of cochains  $\varphi \otimes \varphi'$  to the form denoted again  $\varphi \otimes \varphi'$  defined by  $(\varphi \otimes \varphi')(a \otimes b) = (-1)^{|\varphi'| |a|} \varphi(a) \varphi'(b)$ . Then

(1) there exists  $h \in \text{Ext}_{(C_*(M) \otimes C_*(M'))^\vee}^{m+m'-n-n'}((C_*N \otimes C_*N')^\vee, (C_*M \otimes C_*M')^\vee)$  such that in the derived category of  $C^*(M \times M')$ -modules

$$\begin{array}{ccc} C^*(N \times N') & \xrightarrow{(f \times f')^!} & C^*(M \times M') \\ EZ^\vee \downarrow & & \downarrow EZ^\vee \\ (C_*(N) \otimes C_*(N'))^\vee & \xrightarrow[h]{} & (C_*(M) \otimes C_*(M'))^\vee \end{array}$$

and in the derived category of  $C^*(M) \otimes C^*(M')$ -modules

$$\begin{array}{ccc} (C_*(N) \otimes C_*(N'))^\vee & \xrightarrow{h} & (C_*(M) \otimes C_*(M'))^\vee \\ \Theta \uparrow & & \uparrow \Theta \\ C^*(N) \otimes C^*(N') & \xrightarrow[\varepsilon f^! \otimes f'^!]{} & C^*(M) \otimes C^*(M') \end{array}$$

are commutative squares for some  $\varepsilon \in \mathbb{K}^*$ .

(2) Suppose that  $N$ ,  $N'$ ,  $M$  and  $M'$  are Poincaré duality spaces oriented by  $\omega_N \in H^n(N)$ ,  $\omega_{N'} \in H^{n'}(N')$ ,  $\omega_M \in H^m(M)$  and  $\omega_{M'} \in H^{m'}(M')$ . If we orient  $N \times N'$  by  $\omega_N \times \omega_{N'}$  and  $M \times M'$  by  $\omega_M \times \omega_{M'}$  then  $\varepsilon = (-1)^{(m'-n')n}$ .

(3) There exists  $k \in \text{Ext}_{(C_*(E) \otimes C_*(E'))^\vee}^{m+m'-n-n'}((C_*(X) \otimes C_*(X'))^\vee, (C_*(E) \otimes C_*(E'))^\vee)$  such that in the derived category of  $C^*(E \times E')$ -modules

$$\begin{array}{ccc} C^*(X \times X') & \xrightarrow{(g \times g')^!} & C^*(E \times E') \\ EZ^\vee \downarrow & & \downarrow EZ^\vee \\ (C_*(X) \otimes C_*(X'))^\vee & \xrightarrow[k]{} & (C_*(E) \otimes C_*(E'))^\vee \end{array}$$

and in the derived category of  $C^*(E) \otimes C^*(E')$ -modules

$$\begin{array}{ccc} (C_*(X) \otimes C_*(X'))^\vee & \xrightarrow{k} & (C_*(E) \otimes C_*(E'))^\vee \\ \Theta \uparrow & & \uparrow \Theta \\ C^*(X) \otimes C^*(X') & \xrightarrow[\varepsilon g^! \otimes g'^!]{} & C^*(E) \otimes C^*(E') \end{array}$$

are commutative squares.

*Remark 11.7.* Here  $g^! \otimes g'^!$  denotes the  $C^*(E) \otimes C^*(E')$ -linear map defined by

$$(g^! \otimes g'^!)(a \otimes b) = (-1)^{|g^!| |a|} g^!(a) \otimes g'^!(b).$$

Therefore, Theorem 11.6 (2) implies that

$$H^*((g \times g')^!)(a \times b) = (-1)^{(m'-n')(n+|a|)} H^*(g^!)(a) \times H^*(g'^!)(b).$$

The signs of [2, VI.14.3] are different from that mentioned here.

*Proof of Theorem 11.6.* (1) By definition [15, Theorem 1 and p. 449],  $(f \times f')^!$  is a generator of  $\text{Ext}_{C^*(M \times M')}^{\leq m+m'-n-n'}(C^*(N \times N'), C^*(M \times M'))$ . Let  $h$  be the image of

$(f \times f')^!$  by the composite of isomorphisms

$$\begin{array}{c}
\mathrm{Ext}_{C^*(M \times M')}^*(C^*(N \times N'), C^*(M \times M')) \\
\downarrow \cong \mathrm{Ext}_{Id}^*(Id, EZ^\vee) \\
\mathrm{Ext}_{C^*(M \times M')}^*(C^*(N \times N'), (C_*(M) \otimes C_*(M'))^\vee) \\
\uparrow \cong \mathrm{Ext}_{EZ^\vee}^*(EZ^\vee, Id) \\
\mathrm{Ext}_{(C_*(M) \otimes C_*(M'))^\vee}^*((C_*(N) \otimes C_*(N'))^\vee, (C_*(M) \otimes C_*(M'))^\vee).
\end{array}$$

Since  $f^! \otimes f'^!$  is a generator of

$$\begin{aligned}
& \mathrm{Ext}_{C^*(M) \otimes C^*(M')}^{\leq m+m'-n-n'}(C^*(N) \otimes C^*(N'), C^*(M) \otimes C^*(M')) \\
& \cong \mathrm{Ext}_{C^*(M)}^{\leq m-n}(C^*(N), C^*(M)) \otimes \mathrm{Ext}_{C^*(M')}^{\leq m'-n'}(C^*(N'), C^*(M')),
\end{aligned}$$

the image of  $h$  by the composite of isomorphisms

$$\begin{array}{c}
\mathrm{Ext}_{(C_*(M) \otimes C_*(M'))^\vee}^*((C_*(N) \otimes C_*(N'))^\vee, (C_*(M) \otimes C_*(M'))^\vee) \\
\cong \mathrm{Ext}_\Theta^*(\Theta, Id) \\
\mathrm{Ext}_{C^*(M) \otimes C^*(M')}^*(C^*(N) \otimes C^*(N'), (C_*(M) \otimes C_*(M'))^\vee) \\
\cong \mathrm{Ext}_{Id}^*(Id, \Theta) \\
\mathrm{Ext}_{C^*(M) \otimes C^*(M')}^*(C^*(N) \otimes C^*(N'), C^*(M) \otimes C^*(M')).
\end{array}$$

is an element  $\varepsilon(f^! \otimes f'^!)$ , where  $\varepsilon$  is a non-zero constant (2) In cohomology, (1) gives a commutative diagram

$$\begin{array}{ccc}
H^*(N \times N') & \xrightarrow{H^*((f \times f')^!)} & H^*(M \times M') \\
\uparrow \times & & \uparrow \times \\
H^*(N) \otimes H^*(N') & \xrightarrow[\varepsilon H^*(f^!) \otimes H^*(f'^!)]{} & H^*(M) \otimes H^*(M'),
\end{array}$$

where  $\times$  is the cross product. Therefore

$$\begin{aligned}
\omega_M \times \omega_{M'} &= H^*((f \times f')^!)(\omega_N \times \omega_{N'}) = \\
\varepsilon(-1)^{(m'-n')n} H^*(f^!)(\omega_N) \times H^*(f'^!)(\omega_{N'}) &= \varepsilon(-1)^{(m'-n')n} \omega_M \times \omega_{M'}.
\end{aligned}$$

(3) Consider the following cube in the derived category of  $C^*(M \times M')$ -modules

$$\begin{array}{ccccc}
& & (C_*(X) \otimes C_*(X'))^\vee & \xrightarrow{k} & (C_*(E) \otimes C_*(E'))^\vee \\
& \nearrow EZ^\vee & \uparrow (C_*(q) \otimes C_*(q'))^\vee & \nearrow EZ^\vee & \uparrow (C_*(p) \otimes C_*(p'))^\vee \\
C^*(X \times X') & \xrightarrow{\quad} & C^*(E \times E') & \xrightarrow{\quad} & C^*(M \times M') \\
\uparrow C^*(q \times q') & \nearrow EZ^\vee & \uparrow C^*(p \times p') & \nearrow EZ^\vee & \uparrow \\
C^*(N \times N') & \xrightarrow{(f \times f')^!} & C^*(M \times M') & \xrightarrow{\quad} &
\end{array}$$

with  $k$  defined below. Since

$$\begin{aligned}
& \mathrm{Ext}_{C^*(p \times p')}^*(C^*(q \times q'), C^*(E \times E')) : \\
& \mathrm{Ext}_{C^*(E \times E')}^*(C^*(X \times X'), C^*(E \times E')) \rightarrow \mathrm{Ext}_{C^*(M \times M')}^*(C^*(N \times N'), C^*(E \times E'))
\end{aligned}$$

is an isomorphism, it follows that the maps

$$\mathrm{Ext}_{C^*(p \times p')}^*(C^*(q \times q'), (C_*(E) \otimes C_*(E'))^\vee) :$$

$$\begin{aligned} & \mathrm{Ext}_{C^*(E \times E')}^*(C^*(X \times X'), (C_*(E) \otimes C_*(E'))^\vee) \\ & \rightarrow \mathrm{Ext}_{C^*(M \times M')}^*(C^*(N \times N'), (C_*(E) \otimes C_*(E'))^\vee) \end{aligned}$$

and

$$\mathrm{Ext}_{(C_*(p) \otimes C_*(p'))^\vee}^*((C_*(q) \otimes C_*(q'))^\vee, (C_*(E) \otimes C_*(E'))^\vee) :$$

$$\begin{aligned} & \mathrm{Ext}_{(C_*(E) \otimes C_*(E'))^\vee}^*((C_*(X) \otimes C_*(X'))^\vee, (C_*(E) \otimes C_*(E'))^\vee) \\ & \rightarrow \mathrm{Ext}_{(C_*(M) \otimes C_*(M'))^\vee}^*((C_*(N) \otimes C_*(N'))^\vee, (C_*(E) \otimes C_*(E'))^\vee) \end{aligned}$$

are also isomorphisms. Let  $k$  be the image of  $(C_*(p) \otimes C_*(p'))^\vee \circ h$  by the inverse of the isomorphism

$$\mathrm{Ext}_{(C_*(p) \otimes C_*(p'))^\vee}^*((C_*(q) \otimes C_*(q'))^\vee, (C_*(E) \otimes C_*(E'))^\vee).$$

Since  $EZ^\vee \circ (g \times g')!$  and  $k \circ EZ^\vee$  have the same image by

$$\mathrm{Ext}_{C^*(p \times p')}^*(C^*(q \times q'), (C_*(E) \otimes C_*(E'))^\vee),$$

they coincide and hence we have proved the commutativity of the first square in (3). For the second square in (3), the proof is the same using this time the following cube in the derived category of  $C^*(M) \otimes C^*(M')$ -modules

$$\begin{array}{ccccc} & (C_*(X) \otimes C_*(X'))^\vee & \xrightarrow{k} & (C_*(E) \otimes C_*(E'))^\vee & \\ \Theta \swarrow & \uparrow (C_*(q) \otimes C_*(q'))^\vee & & \searrow \Theta & \\ C^*(X) \otimes C^*(X') & \xrightarrow{\varepsilon g^! \otimes g'^!} & C^*(E) \otimes C^*(E') & & (C_*(p) \otimes C_*(p'))^\vee \\ \uparrow C^*(q) \otimes C^*(q') & & \uparrow h & & \uparrow (C_*(p) \otimes C_*(p'))^\vee \\ (C_*(N) \otimes C_*(N'))^\vee & \xrightarrow{\varepsilon f^! \otimes f'^!} & C^*(M) \otimes C^*(M') & \xrightarrow{\Theta} & (C_*(M) \otimes C_*(M'))^\vee \\ \Theta \swarrow & & & \searrow \Theta & \\ C^*(N) \otimes C^*(N') & \xrightarrow{\varepsilon f^! \otimes f'^!} & C^*(M) \otimes C^*(M') & & \end{array}$$

□

We conclude this section with lemmas on maps with non-zero degree in general. These results follow from straightforward calculations.

**Lemma 11.8.** *Let  $A^*$  be a graded vector space and  $\rho : A^* \otimes A^* \rightarrow A^*$  a linear map of degree  $d$ . Put  $\mathbb{A}^* = A^{*-d}$  and define  $m : \mathbb{A}^* \otimes \mathbb{A}^* \rightarrow \mathbb{A}^*$  by*

$$m(a \otimes b) = (-1)^{d(|a|+d)} \rho(a \otimes b),$$

where  $|a| = i$  if  $a \in A^i$ . Then one sees that

- (i) if  $\rho \circ (\rho \otimes 1) = (-1)^d \rho \circ (1 \otimes \rho)$ , then  $m$  is associative, and that
- (ii) if  $\rho \circ T = (-1)^d \rho$  for the switching map  $T : A^* \otimes A^* \rightarrow A^*$ , then

$$m(a \otimes b) = (-1)^{(|a|+d)(|b|+d)} m(b \otimes a)$$

for  $a \otimes b \in \mathbb{A}^* \otimes \mathbb{A}^*$ . In particular, if  $\rho$  satisfies both conditions in (i) and (ii), then  $(\mathbb{A}^*, m)$  is a graded commutative algebra.

**Lemma 11.9.** (i) For a commutative diagram of graded  $\mathbb{K}$ -modules

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ h \downarrow & & \downarrow k \\ C & \xrightarrow{g} & D, \end{array}$$

the square

$$\begin{array}{ccccc} A^\vee & \xleftarrow{f^\vee} & & B^\vee & \\ h^\vee \uparrow & & & & \uparrow k^\vee \\ C^\vee & \xleftarrow{(-1)^{|f||k|+|g||h|} g^\vee} & D^\vee & & \end{array}$$

is commutative.

(ii) Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be maps of graded  $\mathbb{K}$ -modules. Then, the square is commutative:

$$\begin{array}{ccc} A^\vee \otimes C^\vee & \xrightarrow{f^\vee \otimes g^\vee} & B^\vee \otimes D^\vee \\ \cong \downarrow & & \downarrow \cong \\ (A \otimes C)^\vee & \xrightarrow{(f \otimes g)^\vee} & (B \otimes D)^\vee. \end{array}$$

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